A APPENDIX

A.1 Known Results

If $F \in D \xrightarrow{\nearrow} D$ is monotone on the poset $\langle D,\ \sqsubseteq\rangle$ and $a \sqsubseteq F(a)$ then the transfinite iterates are increasing and the limits exits if $\langle D,\ \sqsubseteq\rangle$ is a cpo. However this hypothesis is often two strong since the limits must exist along the iterates not necessarily elsewhere. We use the following fixpoint theorem A.1 (see *e.g.* [?, COROLLARY 3.3]) and fixpoint abstraction theorems.

Theorem A.1 (iterative fixpoint theorem). *Let $f \in \mathcal{L} \xrightarrow{\nearrow} \mathcal{L}$ be a monotone function on a poset $\langle \mathcal{L},\ \sqsubseteq,\ \sqcup\rangle$ (where $\sqcup$ is partially defined). Let ϵ be the least ordinal which cardinality is strictly greater that the cardinality of $\mathcal{L}$. Let $a \in \mathcal{L}$ be such that $a \sqsubseteq f(a)$. Assume the transfinite iterates $\langle f^\delta(a),\ \delta < \epsilon\rangle$ of f from a up to ϵ are well-defined* (i.e. *the lubs do exist* e.g. *on a cpo $\langle \mathcal{L},\ \sqsubseteq,\ \bot\rangle$). Then f has a least fixpoint $\mathsf{lfp}^{\sqsubseteq}_a f = \bigsqcup_{\delta<\epsilon} f^\delta(a)$.*

Hypotheses A.2 (abstraction). Let $\langle \mathcal{C}, \sqsubseteq, \sqcup\rangle$ be a poset and ϵ be the least ordinal which cardinality is strictly greater that the cardinality of $\mathcal{C}$. Let $f \in \mathcal{C} \xrightarrow{\nearrow} \mathcal{C}$ be monotone. Let $a \in \mathcal{C}$ be such that $a \sqsubseteq f(a)$. Assume the transfinite iterates $\langle f^\delta(a),\ \delta < \epsilon\rangle$ of f from a up to ϵ are well-defined. Let $\mathcal{X} \in \wp(\mathcal{C})$ contain the transfinite iterates of f (*i.e.* $\forall \delta < \epsilon\ .\ f^\delta(a) \in \mathcal{X}$).

Let $\langle \mathcal{A},\ \preccurlyeq,\ \curlyvee\rangle$ be poset and ϵ' be the least ordinal which cardinality is strictly greater that the cardinality of $\mathcal{A}$. Let $\overline{f} \in \mathcal{A} \longrightarrow \mathcal{A}$.

Let $\langle \mathcal{X},\ \sqsubseteq\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \mathcal{A},\ \preccurlyeq\rangle$ be a Galois connection.

Theorem A.3 (exact fixpoint abstraction). *Assume Hypotheses A.2, that $\langle \mathcal{X}, \sqsubseteq\rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}}\!\!\!\!\rightarrow \langle \mathcal{A}, \preccurlyeq\rangle$ is a Galois retraction, and the* commutation property $\forall x \in \mathcal{X}\ .\ \alpha \circ f(x) = \overline{f} \circ \alpha(x)$.

Then the transfinite iterates $\langle \overline{f}^{\delta}(\alpha(a)),\ \delta < \min(\epsilon, \epsilon')\rangle$ are well-defined such that $\forall \delta < \min(\epsilon, \epsilon')$. $\alpha(f^\delta(a)) = \overline{f}^{\delta}(\alpha(a))$, and $\alpha(\mathsf{lfp}^{\sqsubseteq}_a f) = \mathsf{lfp}^{\preccurlyeq}_{\alpha(a)} \overline{f} = \displaystyle\bigcurlyvee_{\delta<\min(\epsilon,\epsilon')} \overline{f}^{\delta}(\alpha(a))$.

Theorem A.4 (approximate fixpoint abstraction). *Assume Hypotheses A.2, that $\overline{f} \in \mathcal{A} \longrightarrow \mathcal{A}$ is monotone, that the transfinite iterates $\langle \overline{f}^{\delta}(\alpha(a)),\ \delta < \epsilon'\rangle$ of $\overline{f}$ from $\alpha(a)$ up to ϵ' are well-defined, and the* semi-commutation property $\forall x \in \mathcal{X}\ .\ \alpha(f(x)) \preccurlyeq \overline{f}(\alpha(x))$.

Then $\forall \delta < \min(\epsilon, \epsilon')\ .\ \alpha(f^\delta(a)) \preccurlyeq \overline{f}^{\delta}(\alpha(a))$ and $\mathsf{lfp}^{\sqsubseteq}_a f \sqsubseteq \gamma(\mathsf{lfp}^{\preccurlyeq}_{\alpha(a)} \overline{f})$.

A.2 Calculational design of the meta abstract interpreter of Section 2

Proof. The Jacobi iterates of (2) belong to $\mathcal{X} = \left\{\begin{bmatrix} \bot \bullet [\ell_1^1, h_1^1] \bullet [\ell_1^2, h_1^2] \bullet \ldots \bullet [\ell_1^n, h_1^n] \\ \bot \bullet [\ell_2^1, h_2^1] \bullet [\ell_2^2, h_2^2] \bullet \ldots \bullet [\ell_2^m, h_2^m] \end{bmatrix} \middle| \ n, m \geqslant 0\right\}$. The Jacobi iterates of (3) belong to $\overline{\mathcal{X}} = \left\{\begin{bmatrix} \langle \ell_1,\ h_1\rangle \\ \langle \ell_2,\ h_1\rangle \end{bmatrix} \middle| \ \ell_1, h_1, \ell_2, h_1 \in \mathcal{D}_c\right\}$. We have the Galois connection $\langle \mathcal{X},\ {\preccurlyeq_{\mathrm{pf}}}^2\rangle \underset{\alpha_c^2}{\overset{\gamma_c^2}{\leftrightarrows}} \langle \overline{\mathcal{X}},\ {\sqsubseteq_c}^2\rangle$.

For the semi-commutation condition, let $\overline{X} \in \mathcal{X}$ be an iterate of iterates of (2).

$$\alpha_c^2(\overline{F}(\overline{X}))$$

$$= \begin{bmatrix} \alpha_c(\overline{\bot} \curlyvee \llbracket \overline{X}_1 \bullet ([0,0] \sqcup x) \mathbin{/\!\!/} \overline{X}_2 = \overline{X} \bullet x \rrbracket) \\ \alpha_c(\overline{\bot} \curlyvee \llbracket \overline{X}_2 \bullet (x \oplus [2,2]) \mathbin{/\!\!/} \overline{X}_1 = \overline{X} \bullet x \rrbracket) \end{bmatrix} \qquad \text{(def. } \alpha_c^2\text{)}$$

Let us calculate the first term.

$$\alpha_c(\overline{\bot} \curlyvee \llbracket \overline{X}_1 \bullet ([0,0] \sqcup x) \mathbin{/\!\!/} \overline{X}_2 = \overline{X} \bullet x \rrbracket)$$

$$= \langle \bot_c,\ \bot_c \rangle \sqcup_c^2 \alpha_c(\llbracket \overline{X}_1 \bullet ([0,0] \sqcup x) \mathbin{/\!\!/} \overline{X}_2 = \overline{X} \bullet x \rrbracket)$$

(in a Galois connection, α_c preserves existing joins)

$$= \alpha_c(\llbracket \overline{X}_1 \bullet ([0,0] \sqcup x) \mathbin{/\!\!/} \overline{X}_2 = \overline{X} \bullet x \rrbracket)$$ (def. infimum)

$$= \alpha_c(\llbracket \overline{X}_1 \bullet ([0,0] \sqcup \llbracket m = 0 \mathrel{?} \bot \mathrel{⁏} [\ell_2^m, h_2^m]\rrbracket) \rrbracket)$$

(by def. of the set $\mathcal{X}$ of iterates, $\overline{X}_2$ has the form $\bot \bullet [\ell_2^1, h_2^1] \bullet [\ell_2^2, h_2^2] \bullet \ldots \bullet [\ell_2^m, h_2^m]$ where $m > 0$ and $\overline{X} = \bot \bullet [\ell_2^1, h_2^1] \bullet [\ell_2^2, h_2^2] \bullet \ldots \bullet [\ell_2^{m-1}, h_2^{m-1}]$, or $\overline{X}_2 = \bot$ with $\overline{X} = \ni$ is the empty sequence whenever $m = 0$)

$$= \alpha_c(\overline{X}_1) \sqcup_c^2 \llbracket m = 0 \mathrel{?} \alpha_c([0,0] \sqcup \bot) \mathrel{⁏} \alpha_c([0,0] \sqcup [\ell_2^m, h_2^m])) \rrbracket$$ (def. α_c and conditional)

$$= \llbracket m = 0 \mathrel{?} \alpha_c(\overline{X}_1) \sqcup_c^2 \alpha_c([0,0]) \mathrel{⁏} \alpha_c(\overline{X}_1) \sqcup_c^2 \alpha_c([\min(0, \ell_2^m), \max(0, h_2^m)])) \rrbracket$$

(def. infimum $\bot$, join $\sqcup$ in intervals, and def. conditional)

$$\sqsubseteq_c^2 \llbracket m = 0 \mathrel{?} \alpha_c(\overline{X}_1) \sqcup_c^2 \alpha_c([0,0]) \mathrel{⁏} \alpha_c(\overline{X}_1) \sqcup_c^2 \alpha_c([0,0] \sqcup [\min(0, \ell_2^m), \max(0, h_2^m)])) \rrbracket$$

(since $[0,0] \sqsubseteq [\min(0, \ell_2^m), \max(0, h_2^m)]$ and α_c is increasing)

$$= \llbracket m = 0 \mathrel{?} \alpha_c(\overline{X}_1) \sqcup_c^2 \langle 0,\ 0 \rangle \mathrel{⁏} \alpha_c(\overline{X}_1) \sqcup_c^2 \langle 0,\ 0 \rangle \sqcup_c^2 \alpha_c([\min(0, \ell_2^m), \max(0, h_2^m)])) \rrbracket$$

(α_c preserves existing joins and def. α_c so that $\alpha_c([0,0]) = \langle 0,\ 0 \rangle$)

$$= \alpha_c(\overline{X}_1) \sqcup_c^2 \langle 0,\ 0 \rangle \sqcup_c^2 \llbracket m = 0 \mathrel{?} \langle \bot_c,\ \bot_c \rangle \mathrel{⁏} \alpha_c([\min(0, \ell_2^m), \max(0, h_2^m)])) \rrbracket$$

(factorizing $\alpha_c(\overline{X}_1) \sqcup_c^2 \langle 0,\ 0 \rangle$ in the conditional and $\langle \bot_c,\ \bot_c \rangle$ is the infimum for the lub $\sqcup_c^2$)

$$= \alpha_c(\overline{X}_1) \sqcup_c^2 \langle 0,\ 0 \rangle \sqcup_c^2 \llbracket \langle \min(0, \ell_2),\ \max(0, h_2) \rangle \mathbin{/\!\!/} \alpha_c(\overline{X}_2) = \langle l_2,\ h_2 \rangle \rrbracket$$

(since if $m = 0$ then $\overline{X}_2$ is $\bot$ hence $\alpha_c(\overline{X}_2) = \langle \bot_c, \bot_c \rangle$ so $\langle l_2, h_2 \rangle = \langle \bot_c, \bot_c \rangle$ and therefore $\langle \min(0, \ell_2), \max(0, h_2) \rangle = \langle \bot_c, \bot_c \rangle$ by our convention that $\bot_c$ is absorbent for both min and max)

$$= \llbracket \langle l_1,\ h_1 \rangle \sqcup_c^2 \langle 0,\ 0 \rangle \sqcup_c^2 \langle \min(0, \ell_2),\ \max(0, h_2) \rangle \mathbin{/\!\!/} \alpha_c(\overline{X}_1) = \langle l_1,\ h_1 \rangle, \alpha_c(\overline{X}_2) = \langle l_2,\ h_2 \rangle \rrbracket$$

(def. let construct)

$$= \llbracket \langle l_1 \sqcup_c 0 \sqcup_c \min(0, l_2),\ h_1 \sqcup_c 0 \sqcup_c \max(0, h_2) \rangle \mathbin{/\!\!/} \alpha_c(\overline{X}_1) = \langle l_1,\ h_1 \rangle, \alpha_c(\overline{X}_2) = \langle l_2,\ h_2 \rangle \rrbracket$$

(pairwise def. $\sqcup_c^2$ in $(\mathcal{D}_c)^2$)

$$= F_1^c(\alpha_c(\overline{X}_2), \alpha_c(\overline{X}_2))$$ (def. F_1^c in (3))

Let us calculate the second term.

$$\alpha_c(\overline{\bot} \curlyvee \llbracket \overline{X}_2 \bullet (x \oplus [2,2]) \mathbin{/\!\!/} \overline{X}_1 = \overline{X} \bullet x \rrbracket)$$

$$= \langle \bot_c,\ \bot_c \rangle \sqcup_c^2 \alpha_c(\llbracket \overline{X}_2 \bullet (x \oplus [2,2]) \mathbin{/\!\!/} \overline{X}_1 = \overline{X} \bullet x \rrbracket)$$ (α_c preserves existing joins)

$$= \alpha_c(\llbracket \overline{X}_2 \bullet (x \oplus [2,2]) \mathbin{/\!\!/} \overline{X}_1 = \overline{X} \bullet x \rrbracket)$$ (def. infimum)

$$= \alpha_c(\llbracket n = 0 \mathrel{?} \overline{X}_2 \bullet \bot \mathrel{⁏} \overline{X}_2 \bullet ([\ell_1^n, h_1^n] \oplus [2,2]) \rrbracket)$$

(by def. of the set $\mathcal{X}$ of iterates, $\overline{X}_1$ has the form $\bot \bullet [\ell_1^1, h_1^1] \bullet [\ell_1^2, h_1^2] \bullet \ldots \bullet [\ell_1^n, h_1^n]$ when $n > 0$ and $\overline{X} = \bot \bullet [\ell_1^1, h_1^1] \bullet [\ell_1^2, h_1^2] \bullet \ldots \bullet [\ell_1^{n-1}, h_1^{n-1}]$, or $n = 0$ so $\overline{X}_1 = \bot$ with $\overline{X} = \ni$ is the empty sequence and $\bot \oplus [2,2] = \bot$)

$$= \alpha_c(\overline{X}_2 \bullet \llbracket n = 0 \mathrel{?} \bot \mathrel{⁏} ([\ell_1^n + 2, h_1^n + 2]) \rrbracket)$$ (factoring $\overline{X}_2$ and def. $\oplus$ for intervals)

$$= \alpha_c(\overline{X}_2) \sqcup_c^2 \llbracket n = 0 \mathrel{?} \langle \bot_c,\ \bot_c \rangle \mathrel{⁏} ([\ell_1^n + 2, h_1^n + 2]) \rrbracket)$$ (def. α_c and $\oplus_c$ on $\mathcal{D}_c$)

$= \llparenthesis\, \alpha_c(\overline{X}_2) \sqcup_c^2 \langle \ell_1 \oplus_c 2,\ h_1 \oplus_c 2\rangle \,/\!\!/\, \langle \ell_1,\ h_1\rangle = \alpha_c(\overline{X}_1)\,\rrparenthesis)$

$\wr$since if $n = 0$ then $\overline{X}_1$ is $\perp$ hence $\alpha_c(\overline{X}_1) = \langle \perp_c,\ \perp_c\rangle$ so $\langle l_1,\ h_1\rangle = \langle \perp_c,\ \perp_c\rangle$ and therefore $\langle \ell_1 \oplus_c 2,\ h_1 \oplus_c 2\rangle = \langle \perp_c \oplus_c 2,\ \perp_c \oplus_c 2\rangle \langle \perp_c,\ \perp_c\rangle$ since $\perp_c$ is absorbent for $\oplus_c$$\wr$

$= \llparenthesis\, \langle \ell_2,\ h_2\rangle \sqcup_c^2 \langle \ell_1 \oplus_c 2,\ h_1 \oplus_c 2\rangle \,/\!\!/\, \langle \ell_1,\ h_1\rangle = \alpha_c(\overline{X}_1), \langle \ell_2,\ h_2\rangle = \alpha_c(\overline{X}_2)\,\rrparenthesis)$ $\wr$def. let construct$\wr$

$= \llparenthesis\, \langle l_2 \sqcup_c (l_1 \oplus^c 2),\ h_2 \sqcup_c (h_1 \oplus^c 2)\rangle \,/\!\!/\, \langle \ell_1,\ h_1\rangle = \alpha_c(\overline{X}_1), \langle \ell_2,\ h_2\rangle = \alpha_c(\overline{X}_2)\,\rrparenthesis$ $\wr$pairwise def. $\sqcup_c^2$ in $(\mathcal{D}_c)^2$$\wr$

$= F_2^c(\alpha_c(\overline{X}_1), \alpha_c(\overline{X}_2))$ $\wr$def. F_2^c in (3)$\wr$

Grouping the two terms, we have proved the semi-commutation $\alpha_c^2(\overline{F}(\overline{X})) \mathrel{\dot\sqsubseteq_c^2} F^c(\alpha_c^2(\overline{X}))$. By Theorem A.4, we conclude that $\mathsf{lfp}^{\preccurlyeq_{pf}{}^2}_{\langle\perp,\,\perp\rangle} \overline{F} \preccurlyeq_{pf}{}^2 \alpha_c^2(\mathsf{lfp}^{\dot\sqsubseteq_c^2}_{\langle\langle\perp_c,\,\perp_c\rangle,\,\langle\perp_c,\,\perp_c\rangle\rangle} F^c)$. □

A.3 Proof of Theorem 6.1

Proof. By (a), $x_0 \sqsubseteq \gamma^0(X^0)$. By recurrence $f^k(x_0) \sqsubseteq \gamma^k(X^k)$ for all iterates k since $f^{k+1}(x_0) = f(f^k(x_0)) \sqsubseteq f(\gamma^k(X^k)) \sqsubseteq \gamma^{k+1}(F^k(X^k)) = \gamma^{k+1}(X^{k+1})$ by def. iterates, monotony (b), induction hypothesis, and semi-commutation (c).

— (a) If $\exists n \in \mathbb{N} \,.\, c(f^n(x^0))$, let n_0 be the smallest one. There are two subcases.

— (a.a) If $\exists k \in \mathbb{N} \,.\, C^k(X^k)$. There are two subcases.

– (a.a.a) If $n_0 \geqslant k$ then $x^0 \sqsubseteq \gamma^k(X^k)$ by (d). Assume $f^n(x^0) \sqsubseteq \gamma^k(X^k)$ for $n < n_0$. Then $f^{n+1}(x^0) = f(f^n(x^0)) \sqsubseteq f(\gamma^k(X^k)) \sqsubseteq \gamma^{k+1}(F^k(X^k)) = \gamma^{k+1}(X^k) = \gamma^k(X^k)$ by def. iterates, monotony (b), semi-commutation (c), and stability (d). By recurrence, we conclude $\mathcal{S}[\![P]\!] = f^{n_0}(x^0) \sqsubseteq \gamma^k(X^k)$.

– (a.a.b) Otherwise $n_0 < k$ so by (e), $\mathcal{S}[\![P]\!] = f^{n_0}(x^0) \sqsubseteq f^k(x^0) \sqsubseteq \gamma^k(X^k)$.

— (a.b) Otherwise $\forall k \in \mathbb{N} \,.\, \neg C^k(X^k)$. We have shown $\forall k \in \mathbb{N} \,.\, f^k(x_0) \sqsubseteq \gamma^k(X^k)$ so $\mathcal{S}[\![P]\!] = f^{n_0}(x^0) \sqsubseteq f^\omega(\prod_{k\in\mathbb{N}} f^k(x^0)) \sqsubseteq \gamma^\omega(F^\omega(\prod_{n\in\mathbb{N}} X^k))$ by (f) and (g).

— (b) Otherwise $\forall n \in \mathbb{N} \,.\, \neg c(f^n(x^0))$. There are two subcases.

— (b.a) If $\exists k \in \mathbb{N} \,.\, C^k(X^k)$. Let us prove that $\forall n \in \mathbb{N} \,.\, f^n(x^0) \sqsubseteq \gamma^k(X^k)$. For the basis, we have shown that $\forall n \leqslant k \,.\, f^n(x^0) \sqsubseteq f^k(x^0) \sqsubseteq \gamma^k(X^k)$. For the induction step, $f^{n+1}(x^0) = f(f^n(x^0)) \sqsubseteq f(\gamma^k(X^k)) \sqsubseteq \gamma^{k+1}(F^k(X^k)) = \gamma^{k+1}(X^k) = \gamma^k(X^k)$ by def. iterates, ind. hyp., monotony (b), semi-commutation (c), and (d). So by (h), $\mathcal{S}[\![P]\!] = f^\omega(\prod_{n\in\mathbb{N}} f^n(x^0)) \sqsubseteq \gamma^k(X^k)$.

— (b.b) Otherwise $\forall k \in \mathbb{N} \,.\, \neg C^k(X^k)$. We have shown $\forall l \in \mathbb{N} \,.\, f^k(x^0) \sqsubseteq \gamma^k(X^k)$ so, by (g), $\mathcal{S}[\![P]\!] = f^\omega(\prod_{n\in\mathbb{N}} f^n(x^0)) \sqsubseteq \gamma^\omega(F^\omega(\prod_{n\in\mathbb{N}} X^k))$. □

A.4 Proof of (10)

Proof. Let $\langle X^k,\ k \in \mathbb{N} \cup \{\omega\}\rangle$ be the iterates of the abstract interpreter (5). Let $\langle \overline{X}^k,\ k \in \mathbb{N} \cup \{\omega, \omega+1\}\rangle$ be the transfinite iterates of $\mathcal{F}_{pf}(X^0)$ in (10).

We observe that $\langle \mathcal{D}_{pf}(X^0)(X^0),\ \subseteq,\ \{X^0\},\ \cup\rangle$ is a cpo and that $\mathcal{F}_{pf}(X^0)$ is $\subseteq$-monotone. By definition of the iterates and recurrence, the $\subseteq$-increasing iterates of $\mathcal{F}_{pf}(X^0)$ are $\overline{X}^0 = \{X^0\}$, $\overline{X}^1 = \{X^0, X^0 \bullet X^1\}, \overline{X}^2 = \{X^0, X^0 \bullet X^1, X^0 \bullet X^1 \bullet X^2\}, \ldots, \overline{X}^k = \{X^0, X^0 \bullet X^1, \ldots, X^0 \bullet X^1 \bullet \cdots \bullet X^k\}$. Passing to the limit, we get $\overline{X}^\omega = \bigcup_{k\in\mathbb{N}} \overline{X}^k = \{X^0, X^0 \bullet X^1, X^0 \bullet X^1 \bullet X^2, \ldots, X^0 \bullet X^1 \bullet X^2 \bullet \cdots \bullet X^k \bullet X^{k+1}, \ldots\}$. The next iterate $\overline{X}^{\omega+1} = \mathcal{F}_{pf}(X^0)\, \overline{X}^\omega$ incorporates $X^0 \bullet \ldots \bullet X^k \bullet \cdots \bullet X^\omega$ to get $\mathcal{S}_{pf}[\![\mathrm{A}]\!]\, X^0$ (8). The next iterate is $\overline{X}^{\omega+2} = \mathcal{F}_{pf}(X^0)\, \overline{X}^{\omega+1} = \overline{X}^{\omega+1}$ which is a fixpoint of $\mathcal{F}_{pf}(X^0)$. By theorem A.1, $\overline{X}^{\omega+1}$ is $\mathsf{lfp}^{\subseteq} \mathcal{F}_{pf}(X^0)$. □

A.5 Proof of (12)

Proof. Let $\langle X^k,\ k \in \mathbb{N} \cup \{\omega\}\rangle$ be the iterates of the abstract interpreter (5). Let $\langle \overline{X}^k,\ k \in \mathbb{N} \cup \{\omega, \omega+1\}\rangle$ be the transfinite iterates of $\mathcal{F}_{\mathsf{pf}}(X^0)$ in (10).

We observe that $\langle \mathcal{D}_{\mathsf{pf}}(X^0)(X^0),\ \subseteq,\ \{X^0\},\ \cup\rangle$ is a cpo and that $\mathcal{F}_{\mathsf{pf}}(X^0)$ is $\subseteq$-monotone. By definition of the iterates and recurrence, the $\subseteq$-increasing iterates of $\mathcal{F}_{\mathsf{pf}}(X^0)$ are $\overline{X}^0 = \{X^0\}$, $\overline{X}^1 = \{X^0, X^0 \bullet X^1\}, \overline{X}^2 = \{X^0, X^0 \bullet X^1, X^0 \bullet X^1 \bullet X^2\}, ..., \overline{X}^k = \{X^0, X^0 \bullet X^1, \ldots, X^0 \bullet X^1 \bullet \cdots \bullet X^k\}$. Passing to the limit, we get $\overline{X}^\omega = \bigcup_{k\in\mathbb{N}} \overline{X}^k = \{X^0, X^0 \bullet X^1, X^0 \bullet X^1 \bullet X^2, \ldots, X^0 \bullet X^1 \bullet X^2 \bullet \cdots \bullet X^k \bullet X^{k+1}, \ldots\}$. The next iterate $\overline{X}^{\omega+1} = \mathcal{F}_{\mathsf{pf}}(X^0)\ \overline{X}^\omega$ incorporates $X^0 \bullet \ldots \bullet X^k \bullet \cdots \bullet X^\omega$ to get $\mathcal{S}_{\mathsf{pf}}[\![\mathrm{A}]\!]\, X^0$ (8). The next iterate is $\overline{X}^{\omega+2} = \mathcal{F}_{\mathsf{pf}}(X^0)\ \overline{X}^{\omega+1} = \overline{X}^{\omega+1}$ which is a fixpoint of $\mathcal{F}_{\mathsf{pf}}(X^0)$. By theorem A.1, $\overline{X}^{\omega+1}$ is $\mathsf{lfp}^{\subseteq}\, \mathcal{F}_{\mathsf{pf}}(X^0)$. □

A.6 Proof of (14)

Proof. We apply theorem A.3. For any finite iterate $\overline{X}^k$ of $\mathcal{F}$, hence of the form $\overline{X}^k = \{X^0, X^0 \bullet X^1, \ldots, X^0 \bullet X^1 \bullet \cdots \bullet X^k\}$ for some $k \in \mathbb{N}$, we have

— $\alpha_{\mathsf{m}}(\mathcal{F}(X^0)\ \emptyset)$
$= \alpha_{\mathsf{m}}(\{X^0\})$ $\qquad$ ⟅def. (10) of $\mathcal{F}(X^0)$⟆
$= X^0$ $\qquad$ ⟅definition of α_{m}⟆

which is the first iterate of $\mathsf{lfp}^{\preccurlyeq_{\mathsf{pf}}}_{X^0}\ \mathcal{F}_{\mathsf{m}}(X^0)$.

— $\alpha_{\mathsf{m}}(\mathcal{F}(X^0)\ \overline{X}^k)$
$= \alpha_{\mathsf{m}}(\{X^0\} \cup \{X^0 \bullet \cdots \bullet X^{k'} \bullet X^{k'+1} \mid X^0 \bullet \cdots \bullet X^{k'} \in \overline{X}^k \wedge X^{k'+1} = F^{k'+1}(X^{k'})\})$
⟅definition of $\mathcal{F}$ since the term $\{X^0 \bullet \ldots \bullet X^{k'} \bullet \cdots \bullet X^\omega \mid \forall k' \in \mathbb{N} \,.\, X^0 \bullet \ldots \bullet X^{k'} \in \overline{X}^k \wedge X^\omega = F^\omega(\langle X^{k'},\ k' \in \mathbb{N}\rangle)\}$ is $\emptyset$⟆
$= \alpha_{\mathsf{m}}(\{X^0\}) \curlyvee \alpha_{\mathsf{m}}(\{X^0 \bullet \cdots \bullet X^{k'} \bullet X^{k'+1} \mid X^0 \bullet \cdots \bullet X^{k'} \in \overline{X}^k \wedge X^{k'+1} = F^{k'+1}(X^{k'})\})$
⟅α_{m} preserves existing lubs by the Galois connection $\langle \wp(\mathcal{D}_{\mathsf{pf}}(X^0)),\ \subseteq\rangle \xleftrightarrow[\alpha_{\mathsf{m}}]{\gamma_{\mathsf{m}}} \langle D^{0,+\omega},\ \preccurlyeq_{\mathsf{pf}}\rangle$⟆
$= X^0 \curlyvee \alpha_{\mathsf{m}}(\{X^0 \bullet \cdots \bullet X^{k'} \bullet X^{k'+1} \mid X^0 \bullet \cdots \bullet X^{k'} \in \overline{X}^k \wedge X^{k'+1} = F^{k'+1}(X^{k'})\})$ ⟅definition of α_{m}⟆
$= X^0 \curlyvee \mathsf{let}\ X^0 \bullet \cdots \bullet X^k = \alpha_{\mathsf{m}}(\overline{X}^k) \in D^{0,k+1}\ \mathsf{in}\ X^0 \bullet \cdots \bullet X^k \bullet F^{k+1}(X^k)$
⟅definition of α_{m} and $\overline{X}^k = \{X^0, X^0 \bullet X^1, \ldots, X^0 \bullet X^1 \bullet \cdots \bullet X^k\}$ so that $X^0 \bullet \cdots \bullet X^k \in D^{0,k+1}$ is the longest sequence in $\overline{X}^k$⟆
$= X^0 \curlyvee \mathsf{let}\ \overline{X} = \alpha_{\mathsf{m}}(\overline{X}^k) \in D^{0,k+1}\ \mathsf{in}\ \overline{X} \bullet F^{k+1}(\overline{X}_k)$ $\qquad$ ⟅letting $\overline{X} = X^0 \bullet \cdots \bullet X^k$ so that $\overline{X}_k = X^k$⟆
$= \mathcal{F}_{\mathsf{m}}(\alpha_{\mathsf{m}}(\overline{X}^k))$ $\qquad$ ⟅definition of $\mathcal{F}_{\mathsf{m}}$ in case $\alpha_{\mathsf{m}}(\overline{X}^k) \in D^{0,k+1}$⟆

For the finite iterates $\langle \overline{X}^k,\ k \in \mathbb{N}\rangle$ of $\mathcal{F}$ and $\langle \overline{X}^k,\ k \in \mathbb{N}\rangle$ of $\mathcal{F}_{\mathsf{m}}$, we have $\alpha_{\mathsf{m}}(\overline{X}^0) = \alpha_{\mathsf{m}}(\{X^0\}) = X^0 = \overline{X}^0$ and, by recurrence, using the commutation $\alpha_{\mathsf{m}}(\mathcal{F}(\overline{X}^k)) = \mathcal{F}(\alpha_{\mathsf{m}}(\overline{X}^k))$, we have $\forall k \in \mathbb{N} \,.\, \alpha_{\mathsf{m}}(\overline{X}^k) = \overline{X}^k$.

α_{m} preserves existing lubs by the Galois connection $\langle \wp(\mathcal{D}_{\mathsf{pf}}(X^0)),\ \subseteq\rangle \xleftrightarrow[\alpha_{\mathsf{m}}]{\gamma_{\mathsf{m}}} \langle D^{0,+\omega},\ \preccurlyeq_{\mathsf{pf}}\rangle$ so for the limit of the finite iterates we have $\alpha_{\mathsf{m}}(\overline{X}^\omega) = \alpha_{\mathsf{m}}(\bigcup_{k\in\mathbb{N}} \overline{X}^k) = \displaystyle\curlyvee_{k\in\mathbb{N}} \alpha_{\mathsf{m}}(\overline{X}^k) = \curlyvee_{k\in\mathbb{N}} \overline{X}^k = \overline{X}^\omega$.

For the next transfinite iterates, we have

$\alpha_{\mathsf{m}}(\overline{X}^{\omega+1})$

$= \alpha_m(\mathcal{F}(X^0)\,\overline{X}^{\omega})$ ⟅def. transfinite iterates⟆

$= \alpha_m(\{X^0 \bullet \ldots \bullet X^k \bullet \cdots \bullet X^\omega \mid \forall k \in \mathbb{N} \,.\, X^0 \bullet \ldots \bullet X^k \in \overline{X}^{\omega} \wedge X^\omega = F^\omega(\langle X^k,\ k \in \mathbb{N}\rangle)\})$

⟅def. (10) of $\mathcal{F}(X^0)$ and $\overline{X}^{\omega} = \bigcup_{k\in\mathbb{N}} \overline{X}^k$⟆

$= \mathsf{let}\ \forall k \in \mathbb{N} \,.\, X^0 \bullet \ldots \bullet X^k \in \overline{X}^{\omega}\ \mathsf{in}\ X^0 \bullet \ldots \bullet X^k \bullet \cdots \bullet F^\omega(\langle X^k,\ k \in \mathbb{N}\rangle)$ ⟅def. α_m⟆

$= \alpha_m(\overline{X}^{\omega}) \bullet F^\omega(\langle \alpha_m(\overline{X}^{\omega})_k,\ k \in \mathbb{N}\rangle)$

⟅$\alpha_m(\overline{X}^{\omega}) = \overline{X}^{\omega} = X^0 \bullet \ldots \bullet X^k \bullet \ldots$ so $X^k = \overline{X}^{\omega}_k = \alpha_m(\overline{X}^{\omega})_k$⟆

$= \mathcal{F}_m(X^0)\,(\alpha_m(\overline{X}^{\omega}))$ ⟅def. 12 of $\mathcal{F}_m(X^0)$⟆

$= \mathcal{F}_m(X^0)\,\overline{X}^{\omega}$ ⟅since $\alpha_m(\overline{X}^{\omega}) = \overline{X}^{\omega}$, as shown above⟆

$= \overline{X}^{\omega+1}$ ⟅def. transfinite iterates⟆

We have $\mathcal{F}(X^0)\,(\overline{X}^{\omega+1}) = \overline{X}^{\omega+1} = \mathsf{lfp}^{\subseteq}\,\mathcal{F}(X^0) = \mathcal{S}[\![\mathrm{A}]\!]\,X^0$. Moreover $\mathcal{F}_m(X^0)\,(\overline{X}^{\omega+1}) = \overline{X}^{\omega+1} = \mathsf{lfp}^{\preccurlyeq_{pf}}_{X^0}\,\mathcal{F}_m(X^0)$. It follows that $\alpha_m(\mathcal{S}[\![\mathrm{A}]\!]\,X^0) = \mathsf{lfp}^{\preccurlyeq_{pf}}\,\mathcal{F}_m(X^0)$. □

A.7 Proof of (16)

Proof. Let $\langle \dot{X}^k, k \in \mathbb{N}\cup\{\omega, \omega+1\}\rangle$ be the iterates of $\mathcal{F}_{pr}(X^0)$ from X^0 and $\langle \widehat{X}^k, k \in \mathbb{N}\cup\{\omega, \omega+1\}\rangle$ be the collecting iterates of $\mathcal{T}(X^0)$ from $\{X^0\}$. We prove that $\forall k \in \mathbb{N} \cup \{\omega, \omega+1\} \,.\, \widehat{X}^k = \{\dot{X}^k\}$.

- For the basis, $\widehat{X}^0 = \{X^0\}$ by initialization of the iterates.
- For the induction step,

$\widehat{X}^{k+1}$

$= \{\mathcal{F}_{pr}(X^0)\,\dot{X} \mid \dot{X} \in \widehat{X}^k\}$ ⟅def. iterates⟆

$= \{\mathcal{F}_{pr}(X^0)\,\dot{X} \mid \dot{X} \in \{\dot{X}^k\}\}$ ⟅induction hypothesis⟆

$= \{\mathcal{F}_{pr}(X^0)\,\dot{X}^k\}$ ⟅def. $\in$⟆

$= \{\dot{X}^{k+1}\}$ ⟅def. iterates⟆

- For the limit, $\dot{X}^\omega = \dot{\bigvee}_{k\in\mathbb{N}} \dot{X}^k$ is the infinite sequence which non-empty prefixes are exactly the $\dot{X}^k$, $k \in \mathbb{N}$. It follows that $\widehat{X}^\omega = \{\dot{X}^\omega\}$ is the $\dot{\precsim}_{pr}$-lub of the $\{\dot{X}^k\}$, $k \in \mathbb{N}$.
- For the next transfinite iterates reaching the fixpoints, we have

$\{\dot{X}^{\omega+1}\}$

$= \{\mathcal{F}_{pr}(X^0)\,\dot{X}^\omega\}$ ⟅def. iterates⟆

$= \{\mathcal{F}_{pr}(X^0)\,\dot{X} \mid \dot{X} \in \{\dot{X}^\omega\}\}$ ⟅def. $\in$⟆

$= \{\mathcal{F}_{pr}(X^0)\,\dot{X} \mid \dot{X} \in \widehat{X}^\omega\}$ ⟅since $\widehat{X}^\omega = \{\dot{X}^\omega\}$⟆

$= \mathcal{T}_{pr}(X^0)\,\widehat{X}^\omega$ ⟅def. (16) of $\mathcal{T}_{pr}(X^0)$⟆

$= \widehat{X}^{\omega+1}$ ⟅def. iterates⟆

Incidentally, observe that the iterates $\langle \dot{X}^k,\ k \in \mathbb{N} \cup \{\omega, \omega+1\}\rangle$ are $\dot{\preccurlyeq}_{pf}$-increasing and so the collecting iterates $\langle \widehat{X}^k,\ k \in \mathbb{N} \cup \{\omega, \omega+1\}\rangle$ are $\dot{\precsim}_{pr}$-increasing. □

A.8 Proof of (18)

Proof. Let $\langle \overline{X}^k, k \in \mathbb{N}\cup\{\omega\}\rangle$ be the iterates of $\mathcal{T}_{pa}$ from $\prod_{i=1}^n \alpha^0_{pa}(X^0_i)$ so that $\overline{X}^0 = \prod_{i=1}^n \alpha^0_{pa}(X^0_i)$, $\overline{X}^{k+1} = \mathcal{T}_{pa}(\overline{X}^k)$, and $\overline{X}^{\omega} = \overline{\bigsqcup_{k\in\mathbb{N}}} \overline{X}^k$. By the iterative fixpoint theorem A.1, their limit is $\overline{X}^{\omega} = \mathsf{lfp}^{\dot{\sqsubseteq}}_{\prod_{i=1}^n \alpha^0_{pa}(X^0_i)}\,\mathcal{T}_{pa}$.

Let $\langle \widehat{X}^k,\ k \in \mathbb{N} \cup \{\omega, \omega+1\}\rangle$ be the collecting iterates of $\mathcal{T}_{\text{pr}}(X^0)$ from $\{X^0\}$, which limit is, by (16), $\widehat{X}^{\omega+1} = \mathcal{C}_{\text{pr}}[\![\text{A}]\!]\, X^0 = \mathsf{lfp}^{\widetilde{\preccurlyeq}_{\text{pr}}}_{\{X^0\}} \mathcal{T}_{\text{pr}}(X^0)$.

Let $\langle \dot{X}^k,\ k \in \mathbb{N} \cup \{\omega, \omega+1\}\rangle$ be the iterates of $\mathcal{F}_{\text{pr}}(X^0)$ from X^0. The proof of (16) shows that $\forall k \in \mathbb{N} \cup \{\omega, \omega+1\}\ .\ \widehat{X}^k = \{\dot{X}^k\}$.

The objective is to show that $\dot{\alpha}_{\text{pa}}(\mathcal{C}_{\text{pr}}[\![\text{A}]\!]\, X^0) = \dot{\alpha}_{\text{pa}}(\mathsf{lfp}^{\widetilde{\preccurlyeq}_{\text{pr}}}_{\{X^0\}} \mathcal{T}_{\text{pr}}(X^0)) \mathrel{\dot{\sqsubseteq}} \mathsf{lfp}^{\dot{\sqsubseteq}}_{\prod_{i=1}^n \alpha^0_{\text{pa}}(X^0_i)} \mathcal{T}_{\text{pa}}$. We apply the approximate fixpoint abstraction theorem A.4 where $\mathcal{X} = \{\widehat{X}^k \mid k \in \mathbb{N} \cup \{\omega, \omega+1\}\}$.

— The initialization condition is

$$\dot{\alpha}_{\text{pa}}(\widehat{X^0}) = \prod_{i=1}^n \overline{\bigsqcup}\{\alpha^j_{\text{pa}}(X^j) \mid X \in \widehat{X}_i \cap D^{0,k} \wedge 0 \leqslant j < k\} = \prod_{i=1}^n \alpha^0_{\text{pa}}(X^0_i)$$

$$\text{\textlbrackdbl def. } \dot{\alpha}_{\text{pa}},\ \widehat{X^0} = \{X^0\},\ X^0 \in D^{0,1}, \text{ def. lub so } \overline{\bigsqcup}\{X\} = X \text{\textrbrackdbl}$$

— The commutation condition is

$$\begin{aligned}
& \dot{\alpha}_{\text{pa}}(\mathcal{T}_{\text{pr}}(X^0)\, \widehat{X}^k) \\
=\ & \dot{\alpha}_{\text{pa}}(\mathcal{T}_{\text{pr}}(X^0)\, \{\dot{X}^k\}) && \text{(as shown in the proof of (16))} \\
=\ & \dot{\alpha}_{\text{pa}}(\{\mathcal{F}_{\text{pr}}(X^0)\, \dot{X} \mid \dot{X} \in \{\dot{X}^k\}\}) && \text{(def. (16) of } \mathcal{T}_{\text{pr}}(X^0)) \\
=\ & \dot{\alpha}_{\text{pa}}(\{\mathcal{F}_{\text{pr}}(X^0)\, \dot{X}^k\}) && \text{(def. } \in) \\
=\ & \dot{\alpha}_{\text{pa}}(\{X^0 \mathbin{\dot{\curlyvee}} \prod_{i=1}^n \dot{X}^k_i \bullet [\![\exists k' \in \mathbb{N} \,.\, \dot{X}^k_i \in D^{0,k'+1} \mathrel{?} F^{k'+1}(\prod_{i=1}^n \dot{X}^k_i(k'))_i \mathrel{:} F^{\omega}(\langle \prod_{i=1}^n \dot{X}^k_i(k'),\ k' \in \mathbb{N}\rangle)_i]\!]\}) \\
& && \text{(def. (14) of } \mathcal{F}_{\text{pr}}(X^0)) \\
=\ & \dot{\alpha}_{\text{pa}}(\{X^0 \mathbin{\dot{\curlyvee}} \prod_{i=1}^n \dot{X}^k_i \bullet [\![k \in \mathbb{N} \mathrel{?} F^{k+1}(\prod_{i=1}^n \dot{X}^k_i(k))_i \mathrel{:} F^{\omega}(\langle \prod_{i=1}^n \dot{X}^k_i(k),\ k \in \mathbb{N}\rangle)_i]\!]\}) \\
& && \text{(the } k\text{-iterate } \dot{X}^k \text{ of } \mathcal{F}_{\text{pr}}(X^0) \text{ belongs to } (D^n)^{0,k+1} \text{ so } k' = k) \\
=\ & \dot{\alpha}_{\text{pa}}(\{\prod_{i=1}^n \dot{X}^k_i \bullet [\![k \in \mathbb{N} \mathrel{?} F^{k+1}(\prod_{i=1}^n \dot{X}^k_i(k))_i \mathrel{:} F^{\omega}(\langle \prod_{i=1}^n \dot{X}^k_i(k),\ k \in \mathbb{N}\rangle)_i]\!]\}) \\
& && \text{(} X^0 \text{ is a prefix of } \dot{X}^k \text{ so } X^0 \mathbin{\dot{\curlyvee}} \dot{X}^k \bullet \cdots = \dot{X}^k \bullet \cdots \text{)} \\
=\ & \prod_{i=1}^n \overline{\bigsqcup}\{\alpha^j_{\text{pa}}(X(j)) \mid X \in \{\dot{X}^k_i \bullet [\![k \in \mathbb{N} \mathrel{?} F^{k+1}(\prod_{i=1}^n \dot{X}^k_i(k))_i \mathrel{:} F^{\omega}(\langle \prod_{i=1}^n \dot{X}^k_i(k),\ k \in \mathbb{N}\rangle)_i]\!]\} \cap \\
& \quad D^{0,k'} \wedge 0 \leqslant j < k'\} && \text{(def. } \dot{\alpha}_{\text{pa}}(\widehat{X}) = \prod_{i=1}^n \overline{\bigsqcup}\{\alpha^j_{\text{pa}}(X^j) \mid X \in \widehat{X}_i \cap D^{0,k'} \wedge 0 \leqslant j < k'\}) \\
=\ & \prod_{i=1}^n \overline{\bigsqcup}\{\alpha^j_{\text{pa}}(\dot{X}^k_i \bullet [\![k \in \mathbb{N} \mathrel{?} F^{k+1}(\prod_{i=1}^n \dot{X}^k_i(k))_i \mathrel{:} F^{\omega}(\langle \prod_{i=1}^n \dot{X}^k_i(k),\ k \in \mathbb{N}\rangle)_i]\!](j)) \mid 0 \leqslant j \leqslant 1+k\} \\
& \quad \text{(either } k \in \mathbb{N} \text{ and } X \in D^{0,k+2} \text{ so } j \in [0, 1+k] \text{ or } k = \omega \text{ and } X \in D^{0,\omega+1} \text{ so } j \in [0,\omega] = \\
& \quad [0, 1+\omega] \text{ so } j \in [0, 1+k] \text{ in both cases)} \\
=\ & \prod_{i=1}^n \overline{\bigsqcup}\{\alpha^j_{\text{pa}}(\dot{X}^k_i(j)) \mid 0 \leqslant j \leqslant k\} \mathbin{\overline{\sqcup}} [\![k \in \mathbb{N} \mathrel{?} \alpha^{k+1}_{\text{pa}}(F^{k+1}(\prod_{i=1}^n \dot{X}^k_i(k))_i) \mathrel{:} \alpha^{\omega}_{\text{pa}}(F^{\omega}(\langle \prod_{i=1}^n \dot{X}^k_i(k), \\
& \quad k \in \mathbb{N}\rangle)_i)]\!] && \text{(def. } \alpha^j_{\text{pa}},\ j \in [0, 1+k])
\end{aligned}$$

$= \prod_{i=1}^{n} \overline{\bigsqcup}\{\alpha_{\mathsf{pa}}^{j}(\dot{X}_i^k(j)) \mid 0 \leqslant j \leqslant k\} \mathbin{\dot{\sqcup}} ⟦ k \in \mathbb{N} \mathrel{?} \prod_{i=1}^{n} \alpha_{\mathsf{pa}}^{k+1}(F^{k+1}(\prod_{i=1}^{n} \dot{X}_i^k(k))_i) \mathrel{⁏} \prod_{i=1}^{n} \alpha_{\mathsf{pa}}^{\omega}(F^{\omega}(\langle \prod_{i=1}^{n} \dot{X}_i^k(k),\ k \in \mathbb{N}\rangle)_i) ⟧$ ⦃pointwise def. $\dot{\sqcup}$ and def. conditional⦄

$= \prod_{i=1}^{n} \overline{\bigsqcup}\{\alpha_{\mathsf{pa}}^{j}(\dot{X}_i^k(j)) \mid 0 \leqslant j \leqslant k\} \mathbin{\dot{\sqcup}} ⟦ k \in \mathbb{N} \mathrel{?} \dot{\alpha}_{\mathsf{pa}}^{k+1}(F^{k+1}(\prod_{i=1}^{n} \dot{X}_i^k(k))) \mathrel{⁏} \dot{\alpha}_{\mathsf{pa}}^{\omega}(F^{\omega}(\langle \prod_{i=1}^{n} \dot{X}_i^k(k), k \in \mathbb{N}\rangle)) ⟧$ ⦃pointwise def. $\dot{\alpha}_{\mathsf{pa}}^{k}$, $k \in \mathbb{N} \cup \{\omega\}$ and $\prod_{i=1}^{n} X_i = X$⦄

$\dot{\sqsubseteq} \prod_{i=1}^{n} \overline{\bigsqcup}\{\alpha_{\mathsf{pa}}^{j}(\dot{X}_i^k(j)) \mid 0 \leqslant j \leqslant k\} \mathbin{\dot{\sqcup}} ⟦ k \in \mathbb{N} \mathrel{?} \overline{F}(\prod_{i=1}^{n} \alpha_{\mathsf{pa}}^{k}(\dot{X}_i^k(k))) \mathrel{⁏} \dot{\bigsqcup_{k \in \mathbb{N}}} \prod_{i=1}^{n} \alpha_{\mathsf{pa}}^{k}(\dot{X}_i^k(k)) ⟧$

⦃by hypotheses $\forall k \in \mathbb{N}. \forall X \in D^k . \dot{\alpha}_{\mathsf{pa}}^{k+1}(F^{k+1}(X)) \mathrel{\dot{\sqsubseteq}} \overline{F}(\dot{\alpha}_{\mathsf{pa}}^{k}(X))$ and $\forall X \in D^{\omega} . \dot{\alpha}_{\mathsf{pa}}^{\omega}(F^{\omega}(\langle X^k, k \in \mathbb{N}\rangle)) \mathrel{\dot{\sqsubseteq}} \dot{\bigsqcup_{k \in \mathbb{N}}} \dot{\alpha}_{\mathsf{pa}}^{k}(X^k)$, pointwise def. $\dot{\alpha}_{\mathsf{pa}}^{k}$, $k \in \mathbb{N}$ and the lub $\dot{\sqcup}$ is monotone⦄

$\dot{\sqsubseteq}\ \dot{\alpha}_{\mathsf{pa}}(\{\dot{X}^k\}) \mathbin{\dot{\sqcup}} ⟦ k \in \mathbb{N} \mathrel{?} \overline{F}(\dot{\alpha}_{\mathsf{pa}}(\{\dot{X}^k\})) \mathrel{⁏} \dot{\alpha}_{\mathsf{pa}}(\{\dot{X}^k\}) ⟧$

⦃$\dot{\alpha}_{\mathsf{pa}}(\{\dot{X}^k\}) = \prod_{i=1}^{n} \overline{\bigsqcup}\{\alpha_{\mathsf{pa}}^{j}(\dot{X}_i^k(j)) \mid \dot{X}_i^k \in D^{0,k+1} \wedge 0 \leqslant j \leqslant k\}$, $\prod_{i=1}^{n} \alpha_{\mathsf{pa}}^{k}(\dot{X}_i^k(k)) \mathrel{\dot{\sqsubseteq}} \dot{\alpha}_{\mathsf{pa}}(\{\dot{X}^k\})$, so $\dot{\bigsqcup_{k \in \mathbb{N}}} \prod_{i=1}^{n} \alpha_{\mathsf{pa}}^{k}(\dot{X}_i^k(k)) \mathrel{\dot{\sqsubseteq}} \dot{\alpha}_{\mathsf{pa}}(\{\dot{X}^k\})$, and $\overline{F}$ and the lub $\dot{\sqcup}$ are monotone⦄

$= ⟦ k \in \mathbb{N} \mathrel{?} \dot{\alpha}_{\mathsf{pa}}(\{\dot{X}^k\}) \mathbin{\dot{\sqcup}} \overline{F}(\dot{\alpha}_{\mathsf{pa}}(\{\dot{X}^k\})) \mathrel{⁏} \dot{\alpha}_{\mathsf{pa}}(\{\dot{X}^k\}) ⟧$ ⦃def. conditional and $X \mathbin{\dot{\sqcup}} X = X$⦄

$= ⟦ k \in \mathbb{N} \mathrel{?} \dot{\alpha}_{\mathsf{pa}}(\widehat{X}^k) \mathbin{\dot{\sqcup}} \overline{F}(\dot{\alpha}_{\mathsf{pa}}(\widehat{X}^k)) \mathrel{⁏} \dot{\alpha}_{\mathsf{pa}}(\widehat{X}^k) ⟧$ ⦃$\forall k \in \mathbb{N} \cup \{\omega, \omega+1\} . \widehat{X}^k = \{\dot{X}^k\}$⦄

$\dot{\sqsubseteq}\ \dot{\alpha}_{\mathsf{pa}}(\widehat{X}^k) \mathbin{\dot{\sqcup}} \overline{F}(\dot{\alpha}_{\mathsf{pa}}(\widehat{X}^k))$ ⦃by def. lub, $X \mathrel{\dot{\sqsubseteq}} X \mathbin{\dot{\sqcup}} Y$⦄

$= \mathcal{T}_{\mathsf{pa}}(\widehat{X}^k)$ ⦃by (18)⦄

— Finally, we have the Galois connection $\langle \mathcal{X}, \subseteq\rangle \underset{\dot{\alpha}_{\mathsf{pa}}}{\overset{\dot{\gamma}_{\mathsf{pa}}}{\leftrightarrows}} \langle \dot{\alpha}_{\mathsf{pa}}(\mathcal{D}_{\mathsf{pa}}^{n}), \dot{\sqsubseteq}\rangle$. □