« Bi-inductive Structural Semantics »

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Contents

1. Motivation

Motivation

– We look for a formalism to specify abstract program semantics

from definitional semantics . . .

to static program analysis algorithms

- coping with termination & non-termination,
- handling the many different styles of presentations found in the literature (rules, fixpoint, equations, constraints, . . .) in a uniform way

– A simple generalization of inductive definitions from sets to posets seems adequate.

2. Bi-inductive Structural Definitions

Over-simplified for the presentation!

Inductive definitions

Set-theoretic [Acz77]

$\langle \wp(\mathcal{U}),\ \subseteq \rangle$	universe
$\dfrac{P}{c} \in \mathcal{R} \quad (P \in \wp(\mathcal{U}), c \in \mathcal{U})$	rules
$F(X) \triangleq \left\{c \;\middle|\; \exists \dfrac{P}{c} \in \mathcal{R} : P \subseteq X\right\}$	transformer
$\mathsf{lfp}^{\subseteq} F \in \wp(\mathcal{U})$	fixpoint def.
$\subseteq\!-\text{least } X : F(X) = X$	equational def.
$\subseteq\!-\text{least } X : F(X) \subseteq X$	constraint def.
$\left\{\dfrac{X}{c} \;\middle|\; X \subseteq \mathcal{U} \wedge c \in F(X)\right\}$	rules

Inductive definitions

Set-theoretic [Acz77]	Order-theoretic	
$\langle \wp(\mathcal{U}),\ \subseteq \rangle$	$\langle \mathcal{D},\ \sqsubseteq \rangle$	universe
$\dfrac{P}{c} \in \mathcal{R} \quad (P \in \wp(\mathcal{U}), c \in \mathcal{U})$	$\dfrac{P}{C} \in \mathcal{R} \quad (P, C \in \mathcal{D})$	rules
$F(X) \triangleq \left\{ c \;\middle|\; \exists \dfrac{P}{c} \in \mathcal{R} : P \subseteq X \right\}$	$F(X) \triangleq \bigsqcup \left\{ C \;\middle|\; \exists \dfrac{P}{c} \in \mathcal{R} : P \sqsubseteq X \right\}$	transformer
$\mathsf{lfp}^{\subseteq} F \in \wp(\mathcal{U})$	$\mathsf{lfp}^{\sqsubseteq} F \in \mathcal{D}$	fixpoint def.
$\subseteq$−least $X : F(X) = X$	$\sqsubseteq$−least $X : F(X) = X$	equational def.
$\subseteq$−least $X : F(X) \subseteq X$	$\sqsubseteq$−least $X : F(X) \sqsubseteq X$	constraint def.
$\left\{ \dfrac{X}{c} \;\middle|\; X \subseteq \mathcal{U} \wedge c \in F(X) \right\}$	$\left\{ \dfrac{X}{F(X)} \;\middle|\; X \in \mathcal{D} \right\}$	rules

3. Semantics of the Eager λ-calculus

Syntax

Syntax of the Eager λ-calculus

$$
\begin{array}{rcll}
\mathsf{x}, \mathsf{y}, \mathsf{z}, \ldots & \in & \mathbb{X} & \text{variables} \\
\mathsf{c} & \in & \mathbb{C} & \text{constants } (\mathbb{X} \cap \mathbb{C} = \emptyset) \\
\mathsf{c} & ::= & \mathsf{0} \mid \mathsf{1} \mid \ldots & \\
\mathsf{v} & \in & \mathbb{V} & \text{values} \\
\mathsf{v} & ::= & \mathsf{c} \mid \boldsymbol{\lambda}\, \mathsf{x} \cdot \mathsf{a} & \\
\mathsf{e} & \in & \mathbb{E} & \text{errors} \\
\mathsf{e} & ::= & \mathsf{c}\ \mathsf{a} \mid \mathsf{e}\ \mathsf{a} & \\
\mathsf{a}, \mathsf{a}', \mathsf{a}_1, \ldots, \mathsf{b}, , \ldots & \in & \mathbb{T} & \text{terms} \\
\mathsf{a} & ::= & \mathsf{x} \mid \mathsf{v} \mid \mathsf{a}\ \mathsf{a}' &
\end{array}
$$

Trace Semantics

Example I: Finite Computation

function argument

$((\lambda\, x \cdot x\ x)\ (\lambda\, y \cdot y))\ ((\lambda\, z \cdot z)\ 0)$

$\longrightarrow$ evaluate function

$((\lambda\, y \cdot y)\ (\lambda\, y \cdot y))\ ((\lambda\, z \cdot z)\ 0)$

$\longrightarrow$ evaluate function, cont'd

$(\lambda\, y \cdot y)\ ((\lambda\, z \cdot z)\ 0)$

$\longrightarrow$ evaluate argument

$(\lambda\, y \cdot y)\ 0$

$\longrightarrow$ apply function to argument

0 *a value!*

Example II: Infinite Computation

function argument

$(\boldsymbol{\lambda}\, x \cdot x\ x)\ (\boldsymbol{\lambda}\, x \cdot x\ x)$

$\rightarrow$ apply function to argument

$(\boldsymbol{\lambda}\, x \cdot x\ x)\ (\boldsymbol{\lambda}\, x \cdot x\ x)$

$\rightarrow$ apply function to argument

$(\boldsymbol{\lambda}\, x \cdot x\ x)\ (\boldsymbol{\lambda}\, x \cdot x\ x)$

$\rightarrow$ apply function to argument

$\ldots$ *non termination!*

Example III: Erroneous Computation

function argument

$((\boldsymbol{\lambda}\, \mathsf{x} \cdot \mathsf{x}\ \mathsf{x})\ ((\boldsymbol{\lambda}\, \mathsf{z} \cdot \mathsf{z})\ 0))\ ((\boldsymbol{\lambda}\, \mathsf{y} \cdot \mathsf{y})\ 0)$

$\longrightarrow$ evaluate argument

$((\boldsymbol{\lambda}\, \mathsf{x} \cdot \mathsf{x}\ \mathsf{x})\ ((\boldsymbol{\lambda}\, \mathsf{z} \cdot \mathsf{z})\ 0))\ 0$

$\longrightarrow$ evaluate function

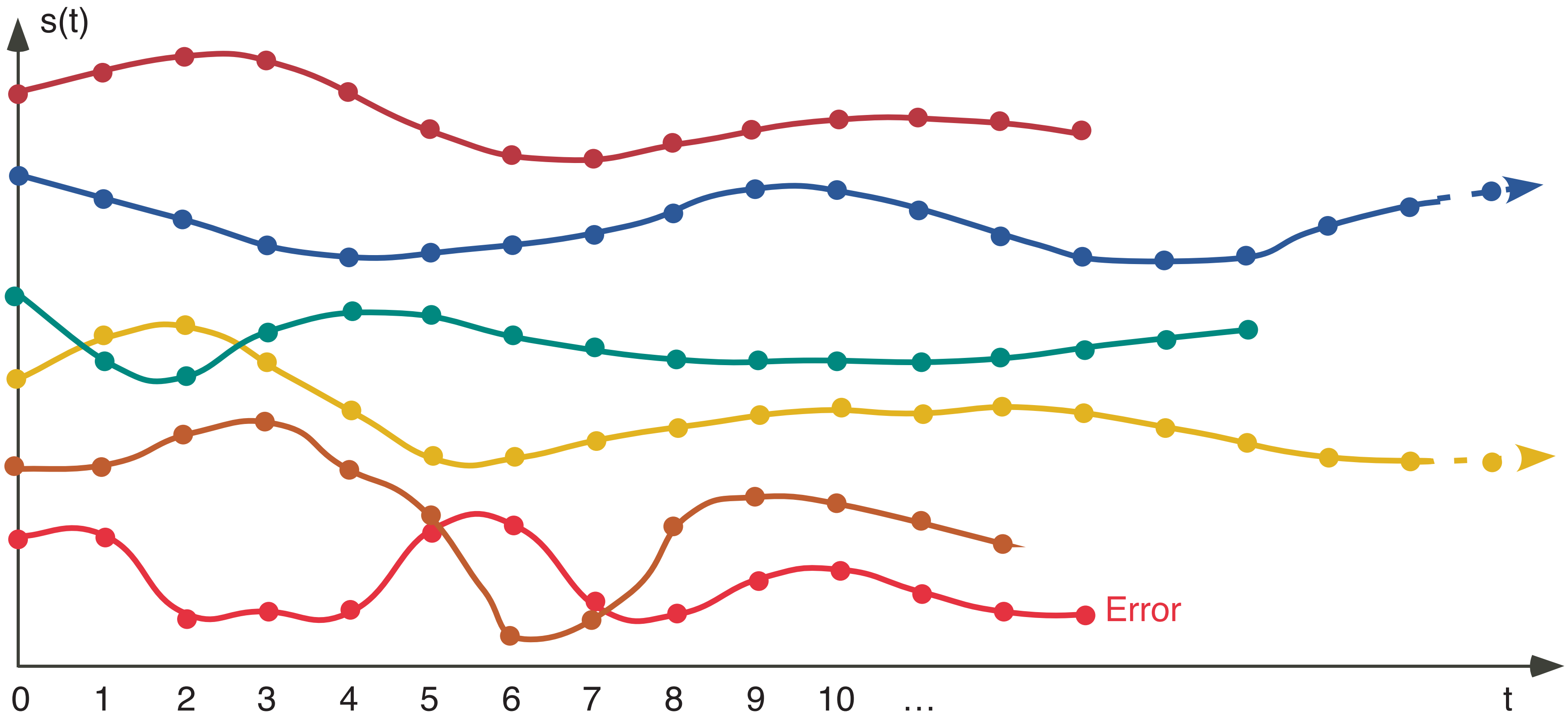
$((\boldsymbol{\lambda}\, \mathsf{x} \cdot \mathsf{x}\ \mathsf{x})\ 0)\ 0$

$\longrightarrow$ evaluate function, cont'd

$(0\ 0)\ 0$

a runtime error!

Finite, Infinite and Erroneous Trace Semantics



Traces

- $\mathbb{T}^\star$ (resp. $\mathbb{T}^+$, $\mathbb{T}^\omega$, $\mathbb{T}^\propto$ and $\mathbb{T}^\infty$) be the set of finite (resp. nonempty finite, infinite, finite or infinite, and nonempty finite or infinite) sequences of terms
- ϵ is the empty sequence $\epsilon \bullet \sigma = \sigma \bullet \epsilon = \sigma$.
- $|\sigma| \in N \cup \{\omega\}$ is the length of $\sigma \in \mathbb{T}^\propto$. $|\epsilon| = 0$.
- If $\sigma \in \mathbb{T}^+$ then $|\sigma| > 0$ and $\sigma = \sigma_0 \bullet \sigma_1 \bullet \ldots \bullet \sigma_{|\sigma|-1}$.
- If $\sigma \in \mathbb{T}^\omega$ then $|\sigma| = \omega$ and $\sigma = \sigma_0 \bullet \ldots \bullet \sigma_n \bullet \ldots$.

Operations on Traces

– For a $\in \mathbb{T}$ and $\sigma \in \mathbb{T}^\infty$, we define $a@\sigma$ to be $\sigma' \in \mathbb{T}^\infty$ such that $\forall i < |\sigma| : \sigma'_i = a\ \sigma_i$

$$\begin{array}{rcl}
\sigma & = & \overset{\sigma_0}{\bullet}\!\!-\!\!-\!\!-\overset{\sigma_1}{\bullet}\!\!-\!\!-\!\!-\overset{\sigma_2}{\bullet}\!\!-\!\!-\!\!-\overset{\sigma_3}{\bullet}\!\!-\!\!- \quad \ldots \quad -\!\!\overset{\sigma_i}{\bullet}\!\!- \quad \ldots \\
a@\sigma & = & \overset{a\ \sigma_0}{\bullet}\!\!-\!\!-\!\!-\overset{a\ \sigma_1}{\bullet}\!\!-\!\!-\!\!-\overset{a\ \sigma_2}{\bullet}\!\!-\!\!-\!\!-\overset{a\ \sigma_3}{\bullet}\!\!-\!\!- \quad \ldots \quad -\!\!\overset{a\ \sigma_i}{\bullet}\!\!- \quad \ldots
\end{array}$$

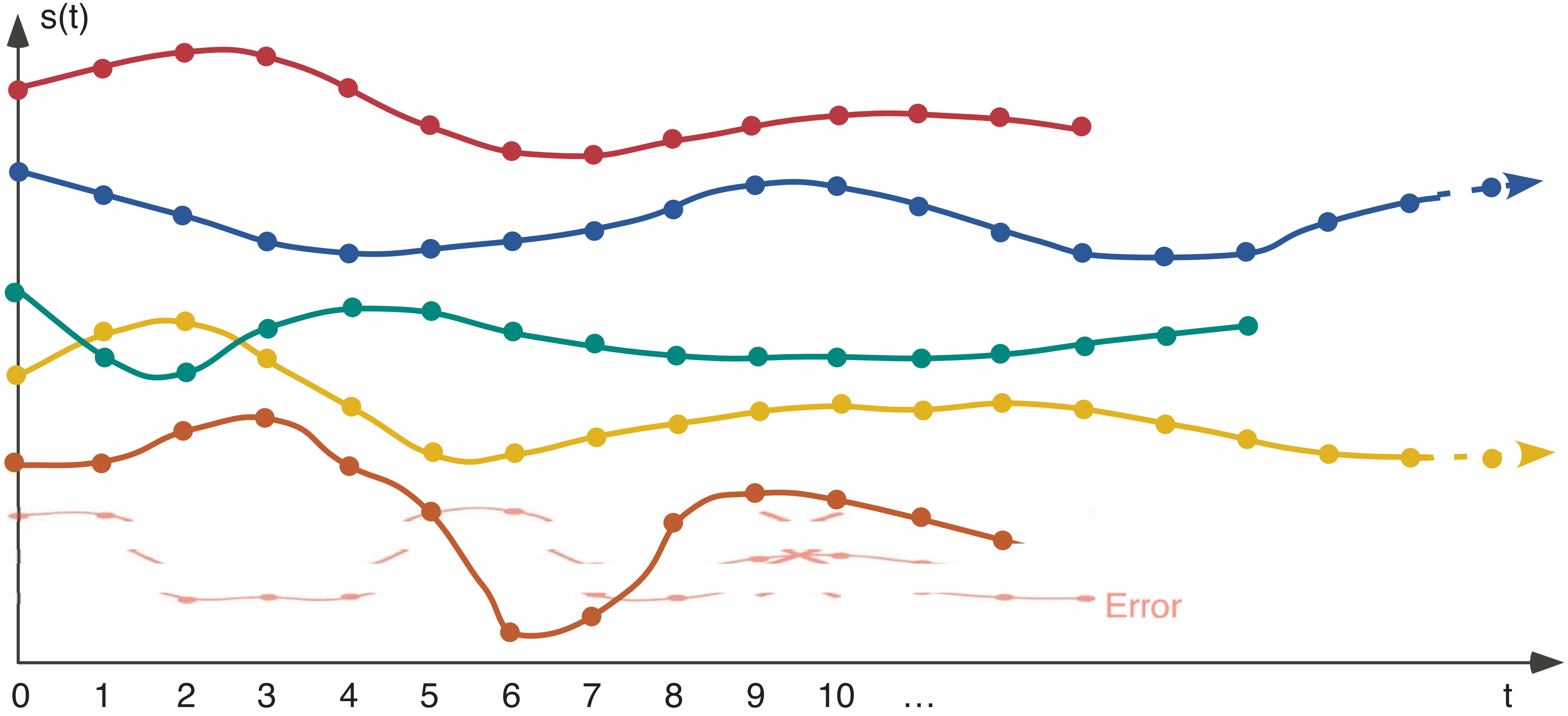
Operations on Traces (Cont'd)

– Similarly for a $\in \mathbb{T}$ and $\sigma \in \mathbb{T}^{\infty}$, $\sigma@a$ is σ' where $\forall i < |\sigma| : \sigma'_i = \sigma_i\ a$

$$\sigma \quad = \quad \sigma_0 \quad \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \ldots \quad \sigma_i \quad \ldots$$

$$\sigma@a \quad = \quad \sigma_0\ a \quad \sigma_1\ a \quad \sigma_2\ a \quad \sigma_3\ a \quad \ldots \quad \sigma_i\ a \quad \ldots$$

Finite and Infinite Trace Semantics



Bifinitary Trace Semantics $\vec{\mathbb{S}}$ of the Eager λ-calculus [1]

$$\mathsf{v} \in \vec{\mathbb{S}},\ \mathsf{v} \in \mathbb{V} \qquad \frac{\mathsf{a}[\mathsf{x} \leftarrow \mathsf{v}] \bullet \sigma \in \vec{\mathbb{S}}}{(\boldsymbol{\lambda}\,\mathsf{x} \cdot \mathsf{a})\ \mathsf{v} \bullet \mathsf{a}[\mathsf{x} \leftarrow \mathsf{v}] \bullet \sigma \in \vec{\mathbb{S}}} \sqsubseteq,\ \mathsf{v} \in \mathbb{V}$$

$$\frac{\sigma \in \vec{\mathbb{S}}^{\omega}}{\mathsf{a}@\sigma \in \vec{\mathbb{S}}} \sqsubseteq,\ \mathsf{a} \in \mathbb{V} \qquad \frac{\sigma \bullet \mathsf{v} \in \vec{\mathbb{S}}^{+},\ (\mathsf{a}\ \mathsf{v}) \bullet \sigma' \in \vec{\mathbb{S}}}{(\mathsf{a}@\sigma) \bullet (\mathsf{a}\ \mathsf{v}) \bullet \sigma' \in \vec{\mathbb{S}}} \sqsubseteq,\ \mathsf{v}, \mathsf{a} \in \mathbb{V}$$

$$\frac{\sigma \in \vec{\mathbb{S}}^{\omega}}{\sigma@\mathsf{b} \in \vec{\mathbb{S}}} \sqsubseteq \qquad \frac{\sigma \bullet \mathsf{v} \in \vec{\mathbb{S}}^{+},\ (\mathsf{v}\ \mathsf{b}) \bullet \sigma' \in \vec{\mathbb{S}}}{(\sigma@\mathsf{b}) \bullet (\mathsf{v}\ \mathsf{b}) \bullet \sigma' \in \vec{\mathbb{S}}} \sqsubseteq,\ \mathsf{v} \in \mathbb{V}$$

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[1] Note: $\mathsf{a}[\mathsf{x} \leftarrow \mathsf{b}]$ is the capture-avoiding substitution of b for all free occurences of x within a. We let $\mathrm{FV}(\mathsf{a})$ be the free variables of a. We define the call-by-value semantics of closed terms (without free variables) $\overline{\mathbb{T}} \triangleq \{\mathsf{a} \in \mathbb{T} \mid \mathrm{FV}(\mathsf{a}) = \varnothing\}$.

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$$\frac{\sigma \in \vec{\mathbb{S}}^{\omega}}{\sigma@\mathsf{b} \in \vec{\mathbb{S}}} \sqsubseteq \qquad\qquad \frac{\sigma \bullet \mathsf{v} \in \vec{\mathbb{S}}^{+}, \quad (\mathsf{v}\ \mathsf{b}) \bullet \sigma' \in \vec{\mathbb{S}}}{(\sigma@\mathsf{b}) \bullet (\mathsf{v}\ \mathsf{b}) \bullet \sigma' \in \vec{\mathbb{S}}} \sqsubseteq, \quad \mathsf{v} \in \mathbb{V}$$

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The Computational Lattice

Given $S, T \in \wp(\mathbb{T}^\infty)$, we define

- $S^+ \triangleq S \cap \mathbb{T}^+$ finite traces
- $S^\omega \triangleq S \cap \mathbb{T}^\omega$ infinite traces
- $S \sqsubseteq T \triangleq S^+ \subseteq T^+ \wedge S^\omega \supseteq T^\omega$ computational order
- $\langle \wp(\mathbb{T}^\infty), \sqsubseteq, \mathbb{T}^\omega, \mathbb{T}^+, \sqcup, \sqcap \rangle$ is a complete lattice

Bifinitary Trace Semantics $\vec{\mathbb{S}}$ of the Eager λ-calculus [1]

$$\mathsf{v} \in \vec{\mathbb{S}}, \quad \mathsf{v} \in \mathbb{V} \qquad\qquad \frac{\mathsf{a}[\mathsf{x} \leftarrow \mathsf{v}] \bullet \sigma \in \vec{\mathbb{S}}}{(\lambda\, \mathsf{x} \cdot \mathsf{a})\ \mathsf{v} \bullet \mathsf{a}[\mathsf{x} \leftarrow \mathsf{v}] \bullet \sigma \in \vec{\mathbb{S}}}\sqsubseteq, \quad \mathsf{v} \in \mathbb{V}$$

$$\frac{\sigma \in \vec{\mathbb{S}}^{\omega}}{\mathsf{a}@\sigma \in \vec{\mathbb{S}}}\sqsubseteq, \quad \mathsf{a} \in \mathbb{V} \qquad\qquad \frac{\sigma \bullet \mathsf{v} \in \vec{\mathbb{S}}^{+}, \ (\mathsf{a}\ \mathsf{v}) \bullet \sigma' \in \vec{\mathbb{S}}}{(\mathsf{a}@\sigma) \bullet (\mathsf{a}\ \mathsf{v}) \bullet \sigma' \in \vec{\mathbb{S}}}\sqsubseteq, \quad \mathsf{v}, \mathsf{a} \in \mathbb{V}$$

$$\frac{\sigma \in \vec{\mathbb{S}}^{\omega}}{\sigma@\mathsf{b} \in \vec{\mathbb{S}}}\sqsubseteq \qquad\qquad \frac{\sigma \bullet \mathsf{v} \in \vec{\mathbb{S}}^{+}, \ (\mathsf{v}\ \mathsf{b}) \bullet \sigma' \in \vec{\mathbb{S}}}{(\sigma@\mathsf{b}) \bullet (\mathsf{v}\ \mathsf{b}) \bullet \sigma' \in \vec{\mathbb{S}}}\sqsubseteq, \quad \mathsf{v} \in \mathbb{V}$$

[1] Note: $\mathsf{a}[\mathsf{x} \leftarrow \mathsf{b}]$ is the capture-avoiding substitution of b for all free occurences of x within a. We let $\mathrm{FV}(\mathsf{a})$ be the free variables of a. We define the call-by-value semantics of closed terms (without free variables) $\overline{\mathbb{T}} \triangleq \{\mathsf{a} \in \mathbb{T} \mid \mathrm{FV}(\mathsf{a}) = \emptyset\}$.

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$$\mathsf{v} \in \vec{\mathbb{S}}, \quad \mathsf{v} \in \mathbb{V}$$

$$\frac{\mathsf{a}[\mathsf{x} \leftarrow \mathsf{v}] \bullet \sigma \in \vec{\mathbb{S}}}{(\boldsymbol{\lambda}\,\mathsf{x} \cdot \mathsf{a})\ \mathsf{v} \bullet \mathsf{a}[\mathsf{x} \leftarrow \mathsf{v}] \bullet \sigma \in \vec{\mathbb{S}}} \sqsubseteq, \quad \mathsf{v} \in \mathbb{V}$$

$$\frac{\sigma \in \vec{\mathbb{S}}^{\omega}}{\mathsf{a}@\sigma \in \vec{\mathbb{S}}} \sqsubseteq, \quad \mathsf{a} \in \mathbb{V}$$

$$\frac{\sigma \bullet \mathsf{v} \in \vec{\mathbb{S}}^{+}, \quad (\mathsf{a}\ \mathsf{v}) \bullet \sigma' \in \vec{\mathbb{S}}}{(\mathsf{a}@\sigma) \bullet (\mathsf{a}\ \mathsf{v}) \bullet \sigma' \in \vec{\mathbb{S}}} \sqsubseteq, \quad \mathsf{v}, \mathsf{a} \in \mathbb{V}$$

$$\frac{\sigma \in \vec{\mathbb{S}}^{\omega}}{\sigma@\mathsf{b} \in \vec{\mathbb{S}}} \sqsubseteq$$

$$\frac{\sigma \bullet \mathsf{v} \in \vec{\mathbb{S}}^{+}, \quad (\mathsf{v}\ \mathsf{b}) \bullet \sigma' \in \vec{\mathbb{S}}}{(\sigma@\mathsf{b}) \bullet (\mathsf{v}\ \mathsf{b}) \bullet \sigma' \in \vec{\mathbb{S}}} \sqsubseteq, \quad \mathsf{v} \in \mathbb{V}$$

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[1] Note: $\mathsf{a}[\mathsf{x} \leftarrow \mathsf{b}]$ is the capture-avoiding substitution of b for all free occurences of x within a. We let $\mathrm{FV}(\mathsf{a})$ be the free variables of a. We define the call-by-value semantics of closed terms (without free variables) $\overline{\mathbb{T}} \triangleq \{\mathsf{a} \in \mathbb{T} \mid \mathrm{FV}(\mathsf{a}) = \varnothing\}$.

Fixpoint big-step maximal trace semantics

The bifinitary trace semantics is

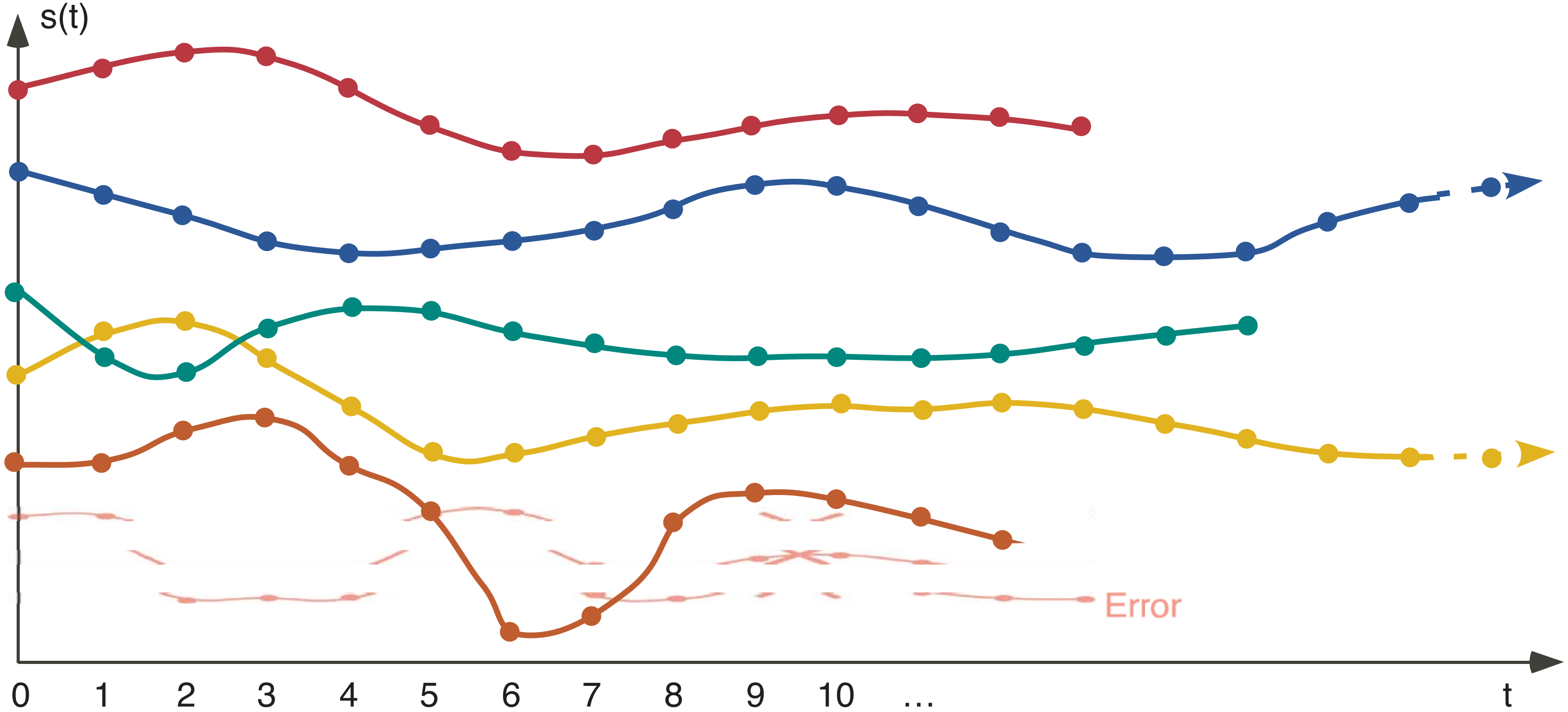
$$\vec{\mathbb{S}} = \mathsf{lfp}^{\sqsubseteq}\ \vec{F}$$

where $\vec{F} \in \wp(\overline{\mathbb{T}}^{\infty}) \mapsto \wp(\overline{\mathbb{T}}^{\infty})$ is

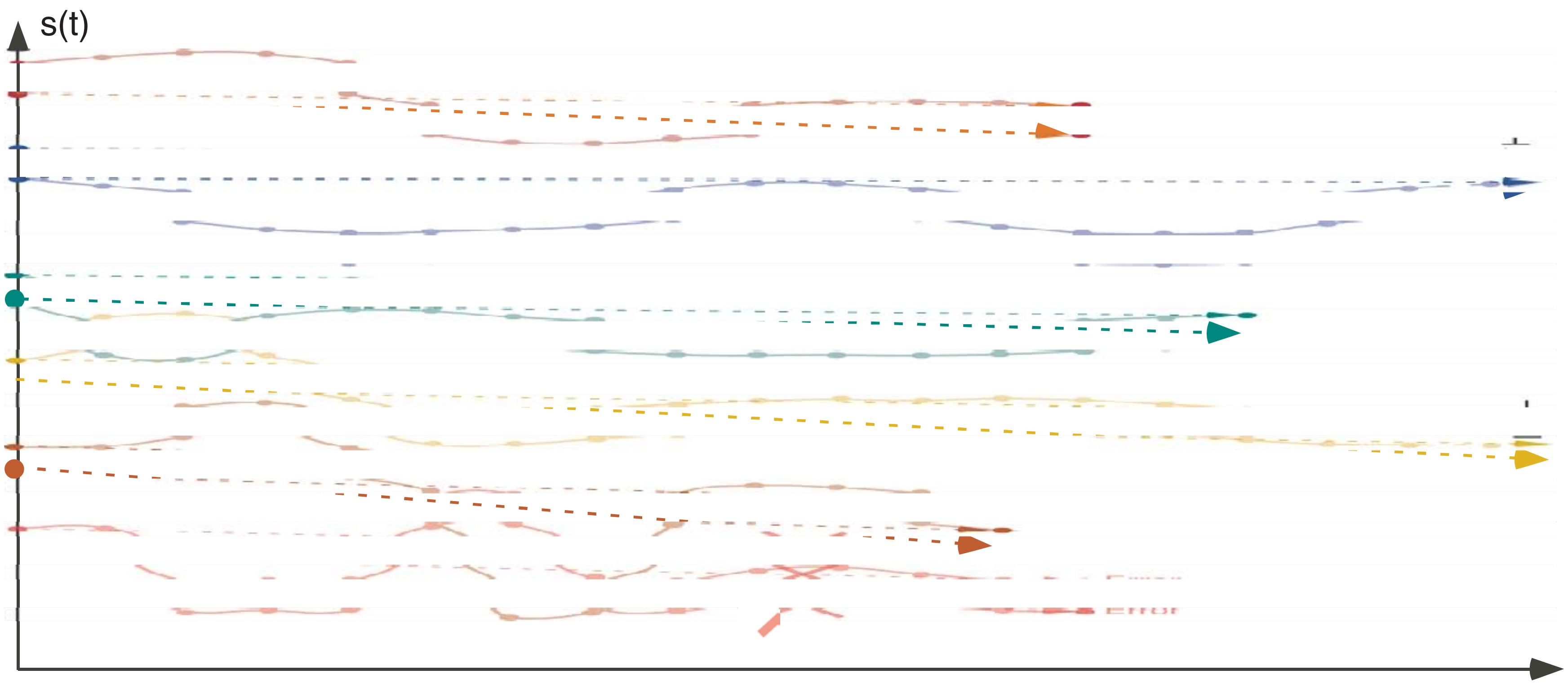
$$\begin{aligned}
\vec{F}(S) \triangleq\ & \{\mathsf{v} \in \overline{\mathbb{T}}^{\infty} \mid \mathsf{v} \in \mathbb{V}\} \cup \\
& \{(\boldsymbol{\lambda}\,\mathsf{x} \cdot \mathsf{a})\ \mathsf{v} \bullet \mathsf{a}[\mathsf{x} \leftarrow \mathsf{v}] \bullet \sigma \mid \mathsf{v} \in \mathbb{V} \wedge \mathsf{a}[\mathsf{x} \leftarrow \mathsf{v}] \bullet \sigma \in S\} \cup \\
& \{\sigma @ \mathsf{b} \mid \sigma \in S^{\omega}\} \cup \\
& \{(\sigma @ \mathsf{b}) \bullet (\mathsf{v}\ \mathsf{b}) \bullet \sigma' \mid \sigma \neq \epsilon \wedge \sigma \bullet \mathsf{v} \in S^{+} \wedge \mathsf{v} \in \mathbb{V} \wedge (\mathsf{v}\ \mathsf{b}) \bullet \sigma' \in S\} \cup \\
& \{\mathsf{a} @ \sigma \mid \mathsf{a} \in \mathbb{V} \wedge \sigma \in S^{\omega}\} \cup \\
& \{(\mathsf{a} @ \sigma) \bullet (\mathsf{a}\ \mathsf{v}) \bullet \sigma' \mid \mathsf{a}, \mathsf{v} \in \mathbb{V} \wedge \sigma \neq \epsilon \wedge \sigma \bullet \mathsf{v} \in S^{+} \wedge (\mathsf{a}\ \mathsf{v}) \bullet \sigma' \in S\}\ .
\end{aligned}$$

Relational Semantics

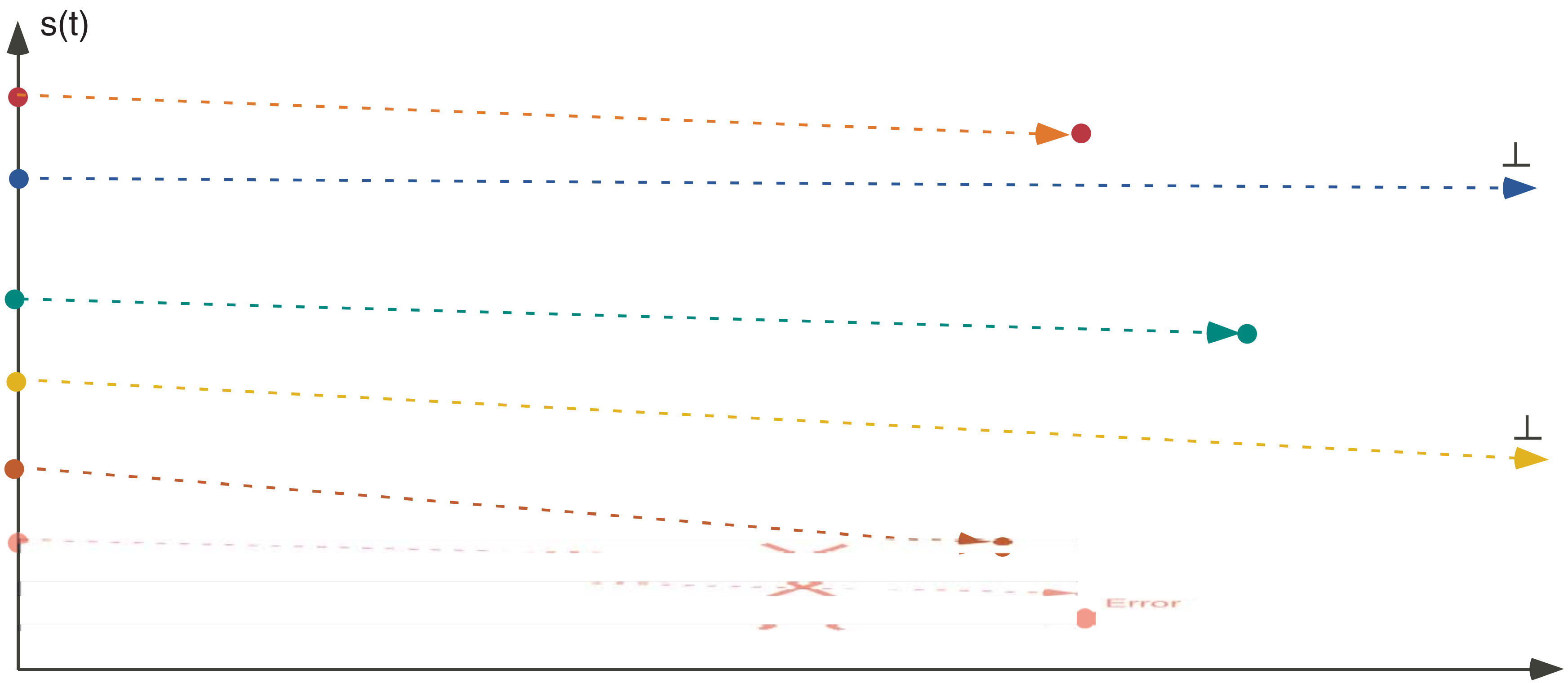
Trace Semantics



Relational Semantics = α(Trace Semantics)



Relational Semantics



Abstraction to the Bifinitary Relational Semantics of the Eager λ-calculus

remember the input/output behaviors,
forget about the intermediate computation steps

$$\begin{array}{rcl}
\alpha(T) & \stackrel{\text{def}}{=} & \{\alpha(\sigma) \mid \sigma \in T\} \\
\alpha(\sigma_0 \bullet \sigma_1 \bullet \ldots \bullet \sigma_n) & \stackrel{\text{def}}{=} & \sigma_0 \Longrightarrow \sigma_n \\
\alpha(\sigma_0 \bullet \ldots \bullet \sigma_n \bullet \ldots) & \stackrel{\text{def}}{=} & \sigma_0 \Longrightarrow \perp
\end{array}$$

Bifinitary Relational Semantics of the Eager λ-calculus

$$\mathsf{v} \Longrightarrow \mathsf{v}, \quad \mathsf{v} \in \mathbb{V}$$

$$\frac{\mathsf{a} \Longrightarrow \bot}{\mathsf{a}\ \mathsf{b} \Longrightarrow \bot}\sqsubseteq \qquad \frac{\mathsf{b} \Longrightarrow \bot}{\mathsf{a}\ \mathsf{b} \Longrightarrow \bot}\sqsubseteq, \quad \mathsf{a} \in \mathbb{V}$$

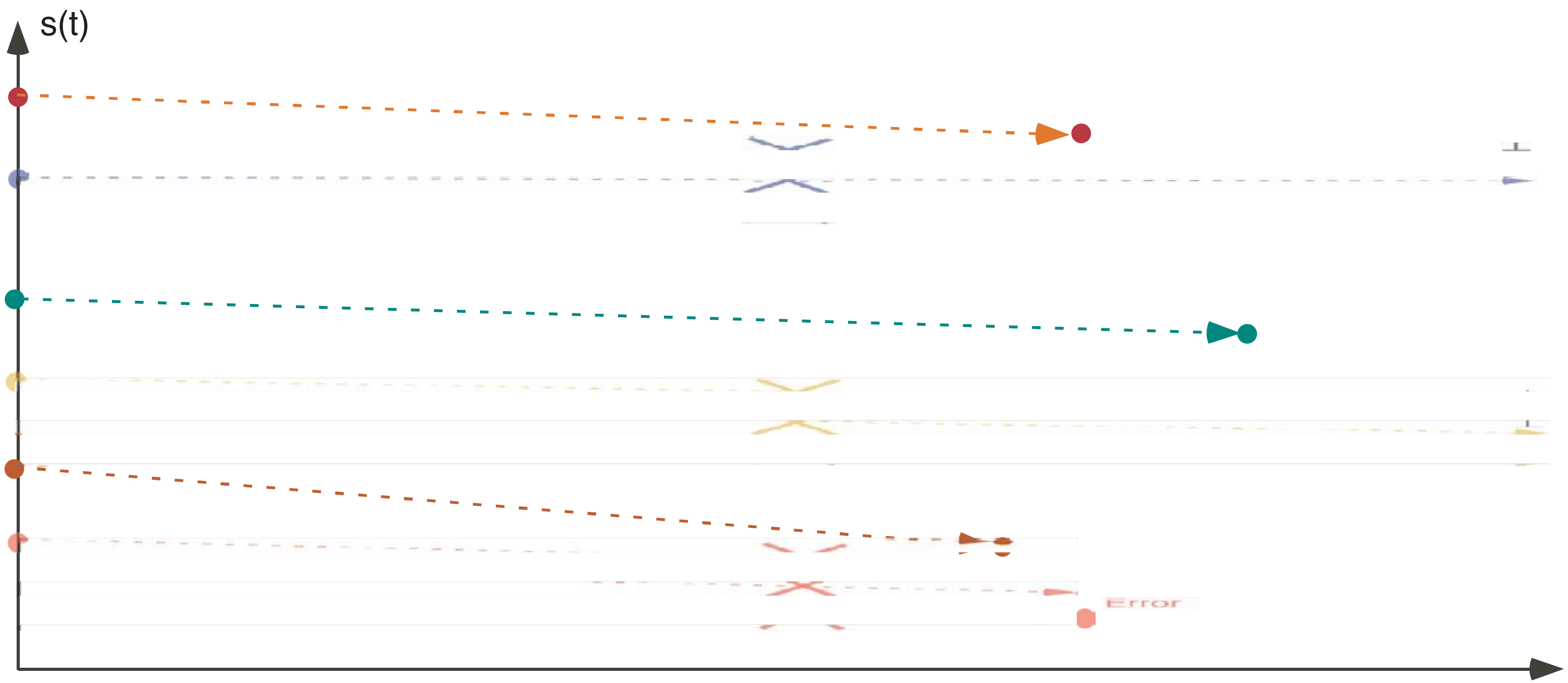
$$\frac{\mathsf{a}[\mathsf{x} \leftarrow \mathsf{v}] \Longrightarrow r}{(\boldsymbol{\lambda}\,\mathsf{x} \cdot \mathsf{a})\ \ \mathsf{v} \Longrightarrow r}\sqsubseteq, \quad \mathsf{v} \in \mathbb{V},\ r \in \mathbb{V} \cup \{\bot\}$$

$$\frac{\mathsf{a} \Longrightarrow \mathsf{v}, \quad \mathsf{v}\ \mathsf{b} \Longrightarrow r}{\mathsf{a}\ \mathsf{b} \Longrightarrow r}\sqsubseteq, \quad \mathsf{v} \in \mathbb{V},\ r \in \mathbb{V} \cup \{\bot\}$$

$$\frac{\mathsf{b} \Longrightarrow \mathsf{v}, \quad \mathsf{a}\ \mathsf{v} \Longrightarrow r}{\mathsf{a}\ \mathsf{b} \Longrightarrow r}\sqsubseteq, \quad \mathsf{a} \in \mathbb{V},\ \mathsf{v} \in \mathbb{V},\ r \in \mathbb{V} \cup \{\bot\}\ .$$

Natural Semantics

Natural Semantics = α(Relational Semantics)



Abstraction to the Natural Big-Step Semantics of the Eager λ-calculus

remember the finite input/output behaviors,
forget about non-termination

$$\alpha(T) \stackrel{\text{def}}{=} \bigcup\{\alpha(\sigma) \mid \sigma \in T\}$$

$$\alpha(\sigma_0 \Longrightarrow \sigma_n) \stackrel{\text{def}}{=} \{\sigma_0 \Longrightarrow \sigma_n\}$$

$$\alpha(\sigma_0 \Longrightarrow \bot) \stackrel{\text{def}}{=} \varnothing$$

Natural Big-Step Semantics of the Eager λ-calculus [Kah88]

$$\mathsf{v} \Longrightarrow \mathsf{v}, \quad \mathsf{v} \in \mathbb{V}$$

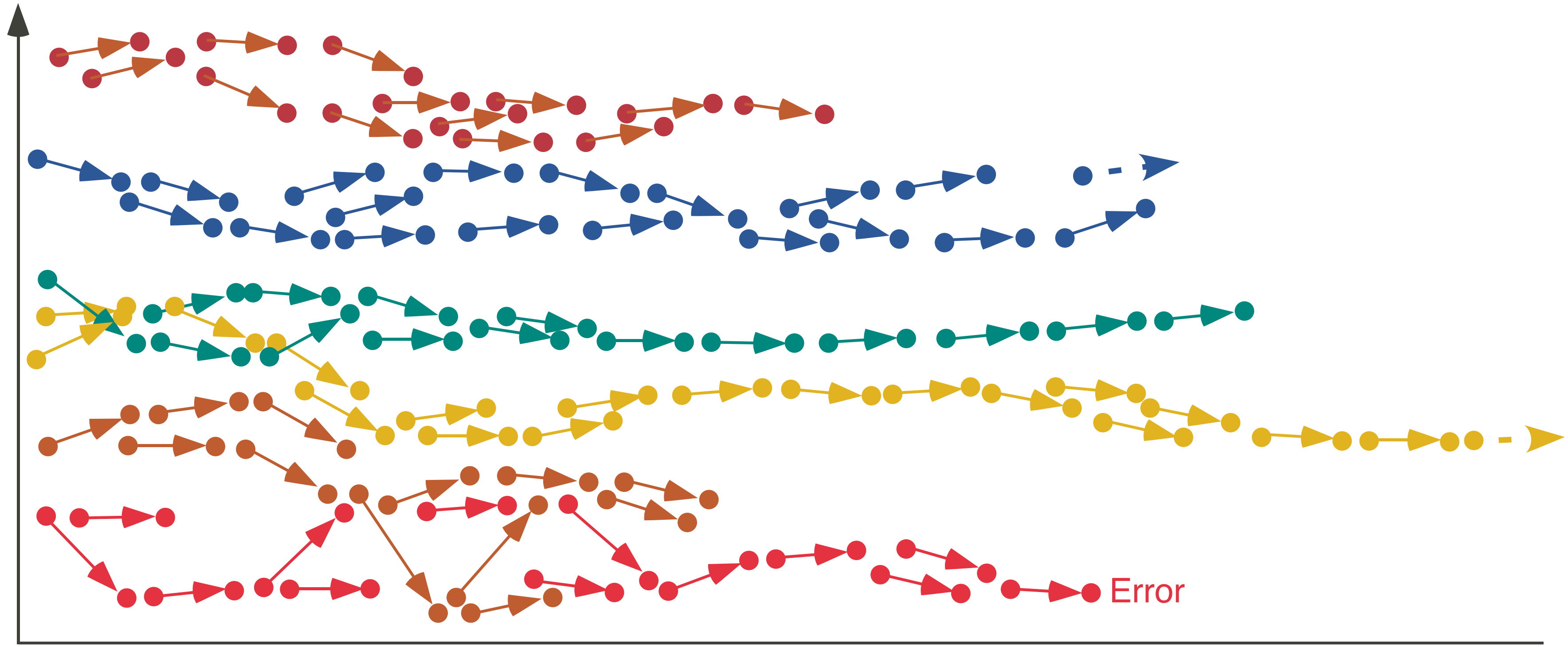
$$\frac{\mathsf{a}[\mathsf{x} \leftarrow \mathsf{v}] \Longrightarrow r}{(\boldsymbol{\lambda}\,\mathsf{x} \cdot \mathsf{a}) \quad \mathsf{v} \Longrightarrow r} \subseteq, \quad \mathsf{v} \in \mathbb{V},\ r \in \mathbb{V}$$

$$\frac{\mathsf{a} \Longrightarrow \mathsf{v}, \quad \mathsf{v}\ \mathsf{b} \Longrightarrow r}{\mathsf{a}\ \mathsf{b} \Longrightarrow r} \subseteq, \quad \mathsf{v} \in \mathbb{V},\ r \in \mathbb{V}$$

$$\frac{\mathsf{b} \Longrightarrow \mathsf{v}, \quad \mathsf{a}\ \mathsf{v} \Longrightarrow r}{\mathsf{a}\ \mathsf{b} \Longrightarrow r} \subseteq, \quad \mathsf{a} \in \mathbb{V},\ \mathsf{v} \in \mathbb{V},\ r \in \mathbb{V}\ .$$

Transition Semantics

Transition Semantics = α(Trace Semantics)



Abstraction to the Transition Semantics of the Eager λ-calculus

remember execution steps,
forget about their sequencing

$$\alpha(T) \stackrel{\text{def}}{=} \bigcup\{\alpha(\sigma) \mid \sigma \in T\}$$

$$\alpha(\sigma_0 \bullet \sigma_1 \bullet \ldots \bullet \sigma_n) \stackrel{\text{def}}{=} \{\sigma_i \longrightarrow \sigma_{i+1} \mid 0 \leqslant i \wedge i < n\}$$

$$\alpha(\sigma_0 \bullet \ldots \bullet \sigma_n \bullet \ldots) \stackrel{\text{def}}{=} \{\sigma_i \longrightarrow \sigma_{i+1} \mid i \geqslant 0\}$$

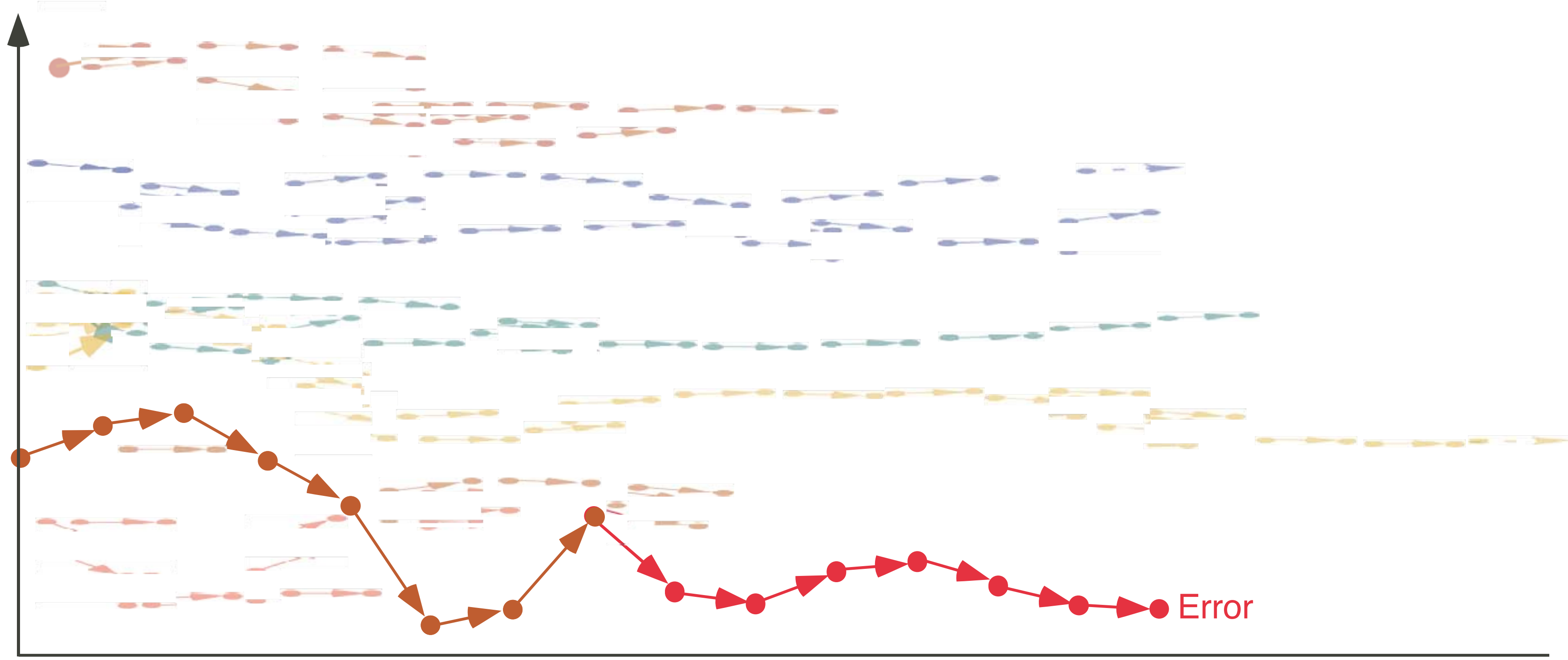
Transition Semantics of the Eager λ-calculus [Plo81]

$$((\boldsymbol{\lambda}\, x \cdot a)\ v) \longrightarrow a[x \leftarrow v]$$

$$\frac{a_0 \longrightarrow a_1}{a_0\ b \longrightarrow a_1\ b} \subseteq$$

$$\frac{b_0 \longrightarrow b_1}{v\ b_0 \longrightarrow v\ b_1} \subseteq .$$

Approximation



$((\boldsymbol{\lambda}\,\mathtt{x}\cdot\mathtt{x}\;\mathtt{x})\;((\boldsymbol{\lambda}\,\mathtt{z}\cdot\mathtt{z})\;0))\;(\boldsymbol{\lambda}\,\mathtt{y}\cdot\mathtt{y}) \longrightarrow ((\boldsymbol{\lambda}\,\mathtt{x}\cdot\mathtt{x}\;\mathtt{x})\;0)\;(\boldsymbol{\lambda}\,\mathtt{y}\cdot\mathtt{y})$
$\longrightarrow (0\;0)\;(\boldsymbol{\lambda}\,\mathtt{y}\cdot\mathtt{y})$ *an error!*

4. Abstraction

Kleenian abstraction

- $\langle \mathcal{D}, \sqsubseteq, \bot, \sqcup \rangle$, $\langle \mathcal{D}^\sharp, \sqsubseteq^\sharp, \bot^\sharp, \sqcup^\sharp \rangle$ dcpos
- $F \in \mathcal{D} \mapsto \mathcal{D}$, $F^\sharp \in \mathcal{D}^\sharp \mapsto \mathcal{D}^\sharp$ monotone
- $\alpha \in \mathcal{D} \mapsto \mathcal{D}^\sharp$ strict and continuous on chains of $\mathcal{D}$
- $\alpha \circ F = F^\sharp \circ \alpha$, commutation condition

$$\Longrightarrow \alpha(\mathsf{lfp}^{\sqsubseteq} F) = \mathsf{lfp}^{\sqsubseteq^\sharp} F^\sharp$$

OK for abstracting finite behaviors, not infinite ones

Tarskian abstraction

- $\langle \mathcal{D}, \sqsubseteq, \bot, \sqcup \rangle$, $\langle \mathcal{D}^\sharp, \sqsubseteq^\sharp, \bot^\sharp, \sqcup^\sharp \rangle$ dcpos
- $F \in \mathcal{D} \mapsto \mathcal{D}$, $F^\sharp \in \mathcal{D}^\sharp \mapsto \mathcal{D}^\sharp$ monotone
- $\alpha \in \mathcal{D} \mapsto \mathcal{D}^\sharp$ preserves meets
- $F^\sharp \circ \alpha \sqsubseteq^\sharp \alpha \circ F$, semi-commutation condition
- $\forall y \in \mathcal{D}^\sharp : (F^\sharp(y) \sqsubseteq^\sharp y) \Longrightarrow (\exists x \in \mathcal{D} : \alpha(x) = y \wedge F(x) \sqsubseteq x$

  $\Longrightarrow \alpha(\mathsf{lfp}^{\sqsubseteq} F) = \mathsf{lfp}^{\sqsubseteq^\sharp} F^\sharp$

OK for abstracting infinite behaviors, not finite ones
$\Rightarrow$ abstract by parts.

5. Conclusion

Conclusion

- Both finite and infinite semantics are needed in static analysis (such as strictness, [Myc80]), typing [Cou97, Ler06], etc;
- Such static analyzes must be proved correct with respect to a semantics chosen at an various level of abstraction (small-step/big-step trace/relational/natural semantics);
- Static analyzes use various equivalent presentations (fixpoints, equational, constraints and inference rules)
- The bifinite extension of SOS should satisfy these needs.

THE END, THANK YOU

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