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An Abstract Interpretation Framework for Termination

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Three principles

Principle I

Program verification methods (formal proof or static analysis methods) are abstract interpretations of a semantics of the programming language (*,**)

(*) P. Cousot and R. Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. *POPL*, 238–252, 1977.

(**) P. Cousot and R. Cousot. Systematic design of program analysis frameworks. *POPL*, 269–282, 1979.

Refinement to principle II

Safety as well as termination verification methods are abstract interpretations of a maximal trace semantics of the programming language

Comments on principle II

- This is well-known for instances of safety (like invariance) using prefix trace semantics(*)
- This is proved in the paper for full safety (omitted in this presentation)
- New for termination

(*) P. Cousot and R. Cousot. Systematic design of program analysis frameworks. *POPL*, 269–282, 1979.

New principle III

More expressive and powerful verification methods are derived by structuring the trace semantics (into a hierarchy of segments)

Comments on principle III

- Syntactic instances have been known for long (different variant functions for nested loops, Hoare logic for total correctness,...)
- Semantic instances have been ignored for long (Burstall's total correctness proof method using intermittent assertions) and very successful recently (Podelski-Rybalchenko)

C. Hoare. An axiomatic basis for computer programming. *Communications of the Association for Computing Machinery*, 12(10):576–580, 1969.

Z. Manna and A. Pnueli. Axiomatic approach to total correctness of programs. *Acta Inf.*, 3:243–263, 1974.

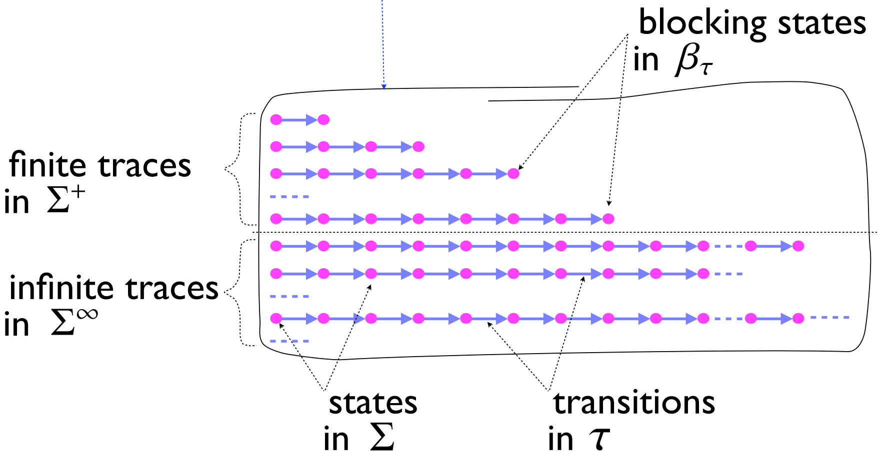
R. Burstall. Program proving as hand simulation with a little induction. *Information Processing*, 308–312. North-Holland, 1974.

A. Podelski and A. Rybalchenko. Transition invariants. *LICS*, 32–41, 2004.

Maximal trace semantics

Maximal trace semantics

- Program $\mathrm{P} \longmapsto \tau^{+\infty}[\![\mathrm{P}]\!] \in \wp(\Sigma^{+\infty})$



Fixpoint maximal trace semantics

- Complete lattice

$$\langle \wp(\Sigma^{*\infty}),\ \sqsubseteq,\ \Sigma^\infty,\ \Sigma^*,\ \sqcup,\ \sqcap \rangle$$

- Computational ordering

$$(T_1 \sqsubseteq T_2) \triangleq (T_1^+ \subseteq T_2^+) \wedge (T_1^\infty \supseteq T_2^\infty) \qquad T^+ \triangleq T \cap \Sigma^+$$
$$(T_1 \sqcup T_2) \triangleq (T_1^+ \cup T_2^+) \cup (T_1^\infty \cap T_2^\infty) \qquad T^\infty \triangleq T \cap \Sigma^\infty$$

- Fixpoint semantics

$$\tau^{+\infty}[\![\mathrm{P}]\!] = \mathrm{lfp}^{\sqsubseteq}_{\Sigma^\infty} \overleftarrow{\phi}^{+\infty}_{\tau}[\![\mathrm{P}]\!]$$
$$= \mathrm{lfp}^{\subseteq}_{\emptyset} \overleftarrow{\phi}^{+}_{\tau}[\![\mathrm{P}]\!] \cup \mathrm{gfp}^{\subseteq}_{\Sigma^\infty} \overleftarrow{\phi}^{\infty}_{\tau}[\![\mathrm{P}]\!]$$
$$\overleftarrow{\phi}^{+\infty}_{\tau}[\![\mathrm{P}]\!]T \triangleq \beta_\tau[\![\mathrm{P}]\!] \sqcup \tau[\![\mathrm{P}]\!] \,;\, T$$

Patrick Cousot, Radhia Cousot: Inductive Definitions, Semantics and Abstract Interpretation. POPL 1992: 83-94

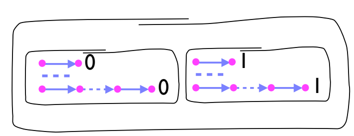
(Trace) properties

Program properties

- A program property P is the set of semantics which have this property:

$$P \in \wp(\wp(\Sigma^{+\infty}))$$

- Example:

$P =$ 

- Strongest property of program P :

$$\{\tau^{+\infty}[\![\mathrm{P}]\!]\}$$

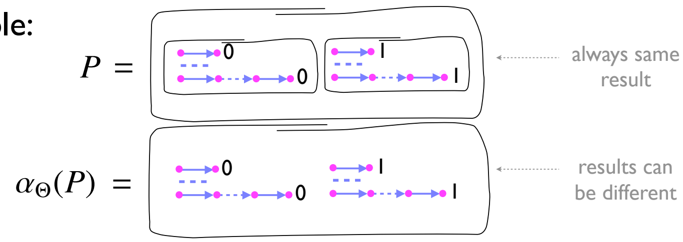
P. Cousot and R. Cousot. Systematic design of program analysis frameworks. *POPL*, 269–282, 1979.

Trace property abstraction

- Trace property abstraction:

$$\alpha_\Theta(P) \triangleq \bigcup P \qquad \langle \wp(\wp(\Sigma^{+\infty})),\ \subseteq\rangle \xleftrightarrow[\alpha_\Theta]{\gamma_\Theta} \langle \wp(\Sigma^{+\infty}),\ \subseteq\rangle$$

- Example:



- The strongest trace property of a trace semantics is this trace semantics $\alpha_\Theta(\{\tau^{+\infty}[\![\mathrm{P}]\!]\}) = \tau^{+\infty}[\![\mathrm{P}]\!]$
- Safety/liveness (termination) are *trace properties*, <u>not</u> general program properties

The Termination Problem

The termination proof problem

- Termination abstraction:

$$\alpha^t(T) \triangleq T \cap \Sigma^+$$

- Termination proof:

$$\alpha^t(\tau^{+\infty}[\![\mathrm{P}]\!]) = \tau^{+\infty}[\![\mathrm{P}]\!]$$

- Termination proofs are not very useful since programs do not *always* terminate

Example

- Arithmetic mean of integers `x` and `y`

```
while (x <> y) {
   x := x - 1;
   y := y + 1
}
```

- Does not *always* terminate e.g.

<x,y> = <1,0> → <0,1> → <-1,2> → <-2,3> → ...

Patrick Cousot: Proving Program Invariance and Termination by Parametric Abstraction, Lagrangian Relaxation and Semidefinite Programming. VMCAI 2005: 1-24

The termination inference problem

- Determine a *necessary* condition for program termination and prove it *sufficient*
- Example:
  - (1) Under which *necessary* conditions

```
while (x <> y) {
   x := x - 1;
   y := y + 1
}
```

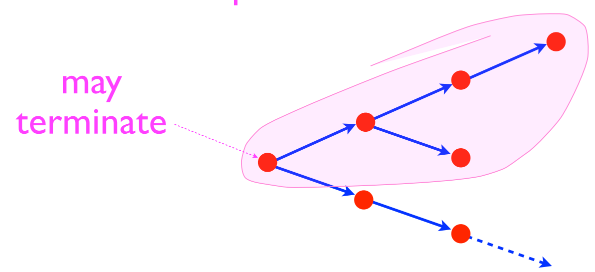
    does terminate?

  - (2) Prove these conditions to be *sufficient*

The Termination Inference Problem

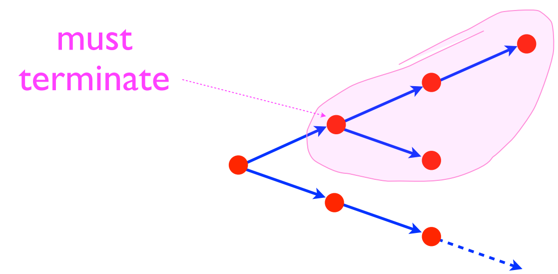
Potential termination

- For non-deterministic programs, we may be interested in potential termination



Definite termination abstraction

- or in definite termination



- Potential and definite termination coincide for deterministic programs. Only definite termination in this presentation.

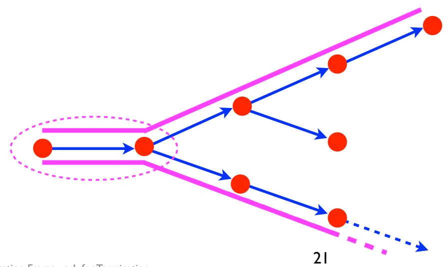
Definite termination trace abstraction

- Prefix Abstraction

$$\mathsf{pf}(\sigma) \triangleq \{\sigma' \in \Sigma^{+\infty} \mid \exists \sigma'' \in \Sigma^{*\infty} : \sigma = \sigma'\sigma''\}$$
$$\mathsf{pf}(T) \triangleq \bigcup \{\mathsf{pf}(\sigma) \mid \sigma \in T\}\,.$$

- Definite termination abstraction

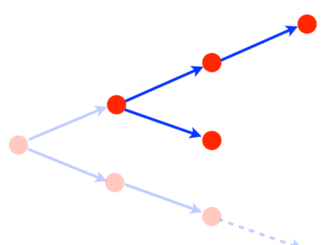
$$\alpha^{\mathsf{Mt}}(T) \triangleq \{\sigma \in T^{+} \mid \mathsf{pf}(\sigma) \cap \mathsf{pf}(T^{\infty}) = \emptyset\}$$



Definite termination

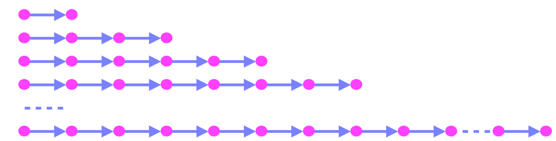
- The semantics/set of traces T *definitely terminates* if and only if

$$\alpha^{\mathsf{Mt}}(T) = T$$

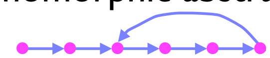


Finite abstractions do not work

- « Abstract and model-check » is *impossible*(*) for termination and *unsound* for non-termination of *unbounded programs*
  - Unbounded executions:



  - Finite homomorphic abstraction:



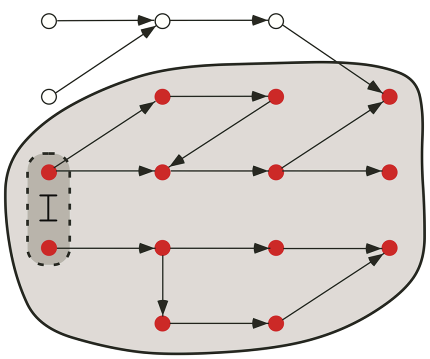
  - Termination: impossible (lasso)
  - Non-termination (lasso): unsound

(*) Excluding trivial solutions, see: Patrick Cousot: Partial Completeness of Abstract Fixpoint Checking. SARA 2000: 1-25

Definite termination domain

Reachability analysis

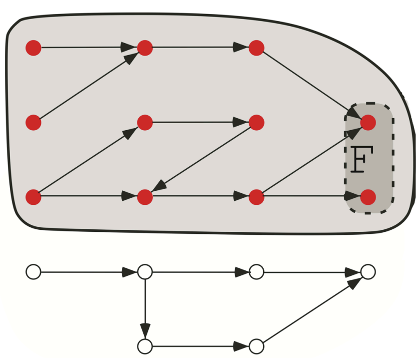
- A forward invariance analysis infers *states potentially reachable from initial states* (by over-approximating an abstract fixpoint $\text{lfp}\ F$)(*)



(*) P. Cousot and R. Cousot. Abstract interpretation: a unified lattice model for static analysis of programs by construction or approximation of fixpoints. *POPL*, 238–252, 1977.

Accessibility analysis

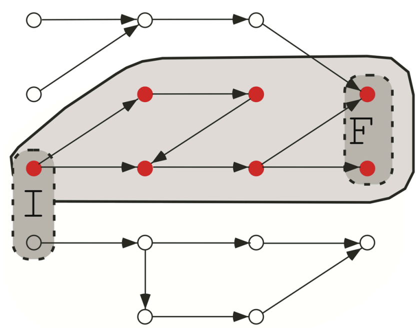
- A backward invariance analysis infers *states potentially / definitely accessing final states* (by over-approximating an abstract fixpoint $\text{lfp}\ B$)(*)



(*) P. Cousot and R. Cousot. Systematic design of program analysis frameworks. *POPL*, 269–282, 1979.

Combined reachability/accessibility analyses

- An iterated forward/backward invariance analysis infers *reachable states potentially/definitely accessing final states* (by over-approximating $\text{lfp}\ F \sqcap \text{lfp}\ B$) (*)



$$
\begin{aligned}
X^0 &= \top \\
&\ldots \\
X^{2n+1} &= \text{lfp}\ \lambda Y \cdot X^{2n} \sqcap F(Y) \\
X^{2n+2} &= \text{lfp}\ \lambda Y \cdot X^{2n+1} \sqcap B(Y) \\
&\ldots
\end{aligned}
$$

(*) P. Cousot. *Méthodes itératives de construction et d'approximation de points fixes d'opérateurs monotones sur un treillis, analyse sémantique de programmes*. Thèse d'État ès sciences math., USMG, Grenoble, 1978.

(*) P. Cousot & R. Cousot. *Abstract interpretation and application to logic programs*. J. Log. Program. 13 (2 & 3): 103–179 (1992)

Example

- Arithmetic mean of two integers `x` and `y`

```
{x>=y}
  while (x <> y){
    {x>=y+2}
       x := x - 1;
    {x>=y+1}
       y := y + 1
    {x>=y}
  }
{x=y}
```

- Necessarily $x \geqslant y$ for proper termination

Example (cont'd)

- Arithmetic mean of two integers x and y (cont'd)

```
while (x <> y) {
    k := k - 1;
    x := x - 1;
    y := y + 1
}
assume (k = 0)
```

auxiliary counter k

Example (cont'd)

- Arithmetic mean of two integers x and y (cont'd)

```
{x=y+2k,x>=y}
  while (x <> y) {
    {x=y+2k,x>=y+2}
      k := k - 1;
    {x=y+2k+2,x>=y+2}
      x := x - 1;
    {x=y+2k+1,x>=y+1}
      y := y + 1
    {x=y+2k,x>=y}
  }
{x=y,k=0}
  assume (k = 0)
{x=y,k=0}
```

auxiliary counter k

- The difference x − y must initially be even for proper termination

Observations

- k provides the *value* of the variant function in the sense of Turing/Floyd
- The constraints on k (hence the variant function) are computed backwards

  $\Longrightarrow$ a *backward* analysis should be able to infer the variant function

R. Floyd. Assigning meaning to programs. *Proc. Symp. in Applied Math.*, Vol. 19, 19–32. Amer. Math. Soc., 1967.

A. Turing. Checking a large routine. *Con. on High Speed Automatic Calculating Machines, Math. Lab., Cambridge, UK*, 67–69, 1949.

The Turing-Floyd termination proof method

R. Floyd. Assigning meaning to programs. *Proc. Symp. in Applied Math.*, Vol. 19, 19–32. Amer. Math. Soc., 1967.

A. Turing. Checking a large routine. *Con. on High Speed Automatic Calculating Machines, Math. Lab., Cambridge, UK*, 67–69, 1949.

The hierarchy of termination semantics

- Maximal trace concrete backward trace semantics

$\downarrow \alpha^{\mathsf{Mt}}$

Definite termination abstract backward trace semantics

$\downarrow \alpha^{\mathsf{w}}$

Weakest pre-condition abstract backward state semantics (termination domain)

$\downarrow \alpha^{\mathsf{rk}}$

Variant function abstract ordinal backward semantics

The ranking abstraction

$$\begin{array}{rcl}
\alpha^{\mathsf{rk}} & \in & \wp(\Sigma \times \Sigma) \mapsto (\Sigma \not\mapsto \mathbb{O}) \\
\alpha^{\mathsf{rk}}(r)s & \triangleq & 0 \quad \text{when} \quad \forall s' \in \Sigma : \langle s,\ s' \rangle \notin r \\
\alpha^{\mathsf{rk}}(r)s & \triangleq & \sup\left\{ \alpha^{\mathsf{rk}}(r)s' + 1 \;\middle|\; \exists s' \in \Sigma : \langle s,\ s' \rangle \in r \;\wedge \right. \\
 & & \left. \forall s' \in \Sigma : \langle s,\ s' \rangle \in r \implies s' \in \mathrm{dom}(\alpha^{\mathsf{rk}}(r)) \right\}
\end{array}$$

- $\alpha^{\mathsf{rk}}(r)$ extracts the well-founded part of relation r
- provides the rank of the elements s in its domain
- strictly decreasing with transitions of relation r

$\Longrightarrow$ the most precise variant function

Fixpoint definition of the variant function

We now apply the abstract interpretation methodology:

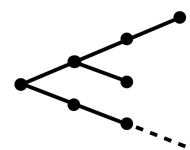
- The maximal trace semantics has a fixpoint definition
- The variant function is an abstraction of the maximal trace semantics
- With this abstraction, we construct a fixpoint definition of the abstract variant semantics

$\Longrightarrow$ *Fixpoint induction* provides a termination proof method

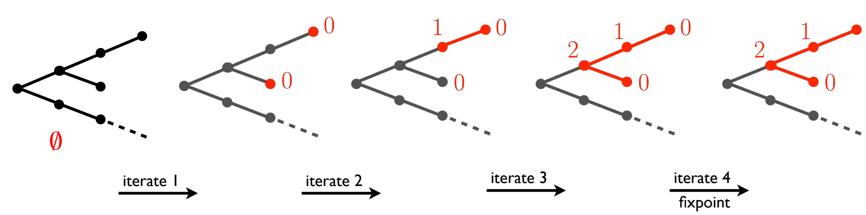
$\Longrightarrow$ Further *abstractions and widenings* provide a static analysis method

Example I

- Maximal trace semantics:



- Ranking fixpoint iterates:



Example II

- Program

```
int x; while (x > 0) { x = x - 2; }
```

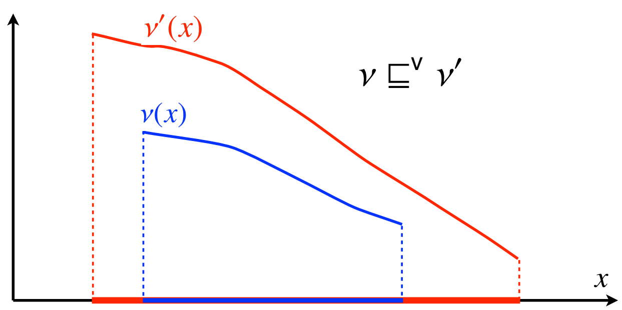
- Fixpoint $\quad \nu \;=\; \mathsf{lfp}^{\sqsubseteq^{\mathsf{v}}}_{\dot{\emptyset}}\; \overleftarrow{\phi}^{\mathsf{Mv}}_{\tau}[\![\mathrm{P}]\!]$

$$\overleftarrow{\phi}^{\mathsf{Mv}}_{\tau}[\![\mathrm{P}]\!](\nu)x \;\triangleq\; \llparenthesis\, x \leqslant 0 \;?\; 0 \;:\; \sup\{\,\nu(x-2)+1 \mid x-2 \in \mathsf{dom}(\nu)\}\,\rrparenthesis$$

- Iterates

$$\begin{aligned}
\nu^0 &= \dot{\emptyset}\\
\nu^1 &= \lambda x \in [-\infty, 0] \bullet 0\\
\nu^2 &= \lambda x \in [-\infty, 0] \bullet 0 \,\dot{\cup}\, \lambda x \in [1, 2] \bullet 1\\
\nu^3 &= \lambda x \in [-\infty, 0] \bullet 0 \,\dot{\cup}\, \lambda x \in [1, 2] \bullet 1 \,\dot{\cup}\, \lambda x \in [3, 4] \bullet 2\\
&\ldots\\
\nu^n &= \lambda x \in [-\infty, 0] \bullet 0 \,\dot{\cup}\, \lambda x \in [1, 2 \times (n-1)] \bullet (x+1) \div 2\\
&\ldots\\
\nu^\omega &= \lambda x \in [-\infty, 0] \bullet 0 \,\dot{\cup}\, \lambda x \in [1, +\infty] \bullet (x+1) \div 2\,.
\end{aligned}$$

Computational order on functions



$$\nu \sqsubseteq^{\mathsf{v}} \nu' \quad\triangleq\quad \mathsf{dom}(\nu) \subseteq \mathsf{dom}(\nu') \wedge \forall x \in \mathsf{dom}(\nu) : \nu(x) \preccurlyeq \nu'(x)$$

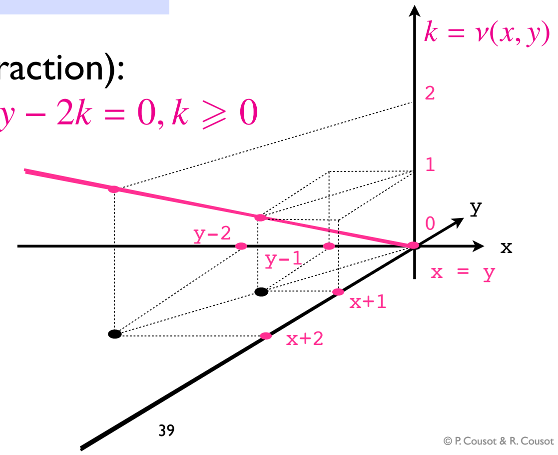
Example III

- Program:

```
{ even(x-y), x >= y }
while (x <> y) {
   x := x - 1;
   y := y + 1
}
{ x = y }
```

- Iterates (linear abstraction):

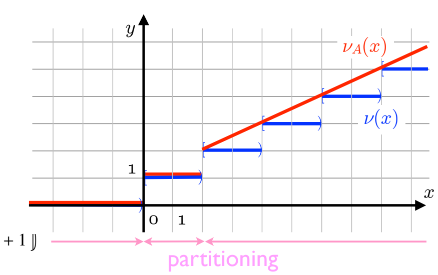
$\exists k : \nu(x, y) = k,\, x - y - 2k = 0,\, k \geqslant 0$



Example IV

- In general a widening is needed to enforce convergence
- Program: `int x; while (x > 0) { x = x - 2; }`
- Iterates with widening:

$$\begin{aligned}
\nu_A^0 &= \lambda x \in [-\infty, +\infty] \bullet \bot\\
\nu_A^1 &= \lambda x \bullet \llparenthesis\, x \in [-\infty, 0] \;?\; 0 \;:\; x \in [1, +\infty] \;?\; \bot \,\rrparenthesis\\
\nu_A^2 &= \lambda x \in [-\infty, 0] \bullet 0 \,\dot{\cup}\, \lambda x \in [1, 2] \bullet 1 \,\dot{\cup}\, \lambda x \in [3, +\infty] \bullet \bot\\
\nu'^3_A &= \lambda x \bullet \llparenthesis\, x \in [-\infty, 0] \;?\; 0 \;:\; x \in [1, 2] \;?\; 1 \;:\; x \in [3, 4] \;?\; 2\\
&\qquad\quad : x \in [5, +\infty] \;?\; \bot \,\rrparenthesis\\
\nu_A^3 &= \nu_A^2 \;\dot{\triangledown}^{\mathsf{v}}\; \nu'^3_A\\
\nu'^4_A &= \lambda x \bullet \llparenthesis\, x \in [-\infty, 0] \;?\; 0 \;:\; x \in [1, 2] \;?\; 1 \;:\; x \in [3, +\infty] \;?\; \frac{x}{2} + 1 \,\rrparenthesis\\
\nu_A^4 &= \nu_A^3\,.
\end{aligned}$$



Objection I: Turing/Floyd's method goes forward not backward!

- An analysis can be inverted using auxiliary variables(*)

```
        int x;
        while (c(x)) {
before ···>
          x := f(x)
after  ···>
        }
```

→

```
int x, x0;
while (c(x)) {
   x0 := x;
   x := f(x)
}
```

Backward variant ν:

$\nu(x_{before}) = \nu(x_{after}) + 1$

$\Longleftrightarrow \nu(x_{before}) = \nu(f(x_{before})) + 1$

Forward variant ν:

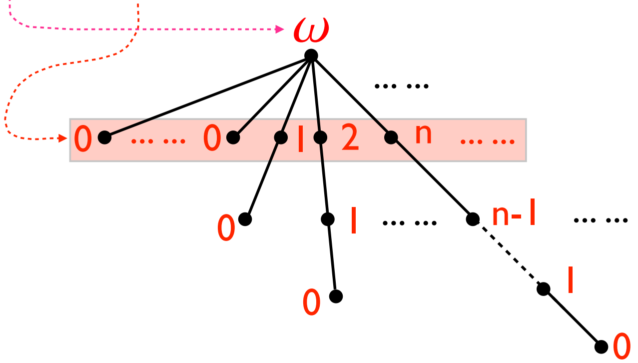
$\nu(x_0) = \nu(x) + 1$

$\Longleftrightarrow \nu(x_0) = \nu(f(x_0)) + 1$

(*) P. Cousot. Semantic foundations of program analysis. *Program Flow Analysis: Theory and Applications*, ch. 10, 303–342. Prentice-Hall, 1981.

Objection II: you need ordinals! (*)

- Example: `x := ?; while (x >= 0) do x := x - 1 od`
- Ranking:



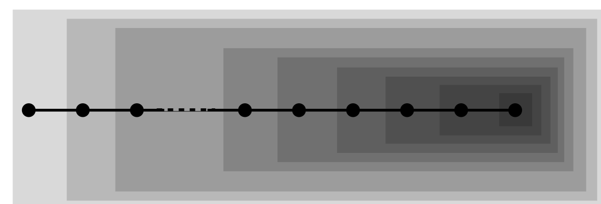
- To avoid transfinite ordinals/well-founded orders(*) for unbounded non-determinism, the computations need to be structured!

(*) R. Floyd. Assigning meaning to programs. *Proc. Symp. in Applied Math.*, Vol. 19, 19–32. Amer. Math. Soc., 1967.

Structuring trace semantics with segments

Floyd/Turing termination proof method

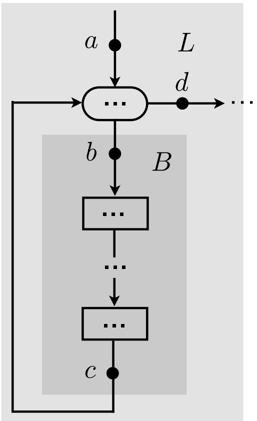
- Trivial postfix structuring of traces into segments



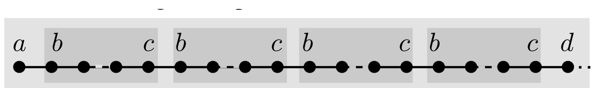
- Also used for termination of straight-line code (no need for variant functions)

Floyd with nested loops

- The trace semantics is recursively structured in segments according to loop nesting



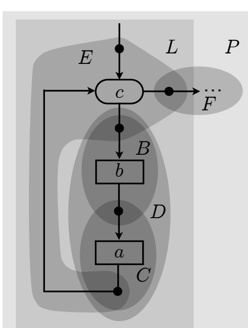
Prove termination of outer loop assuming termination of body/ nested inner loops



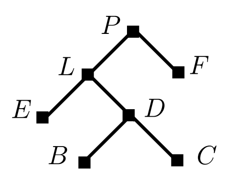
(equivalent to lexicographic orderings)

Hoare logic

- The trace semantics is recursively structured in segments according to the program syntax
- `while (c) { b; a }...`



tree structure of the segmentation:

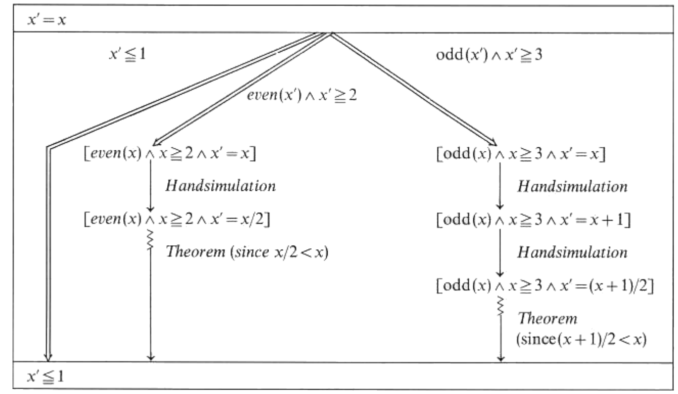


$\{P, PF, PL, PLE, PLD, PLDB, PLDC\}$

C. Hoare. An axiomatic basis for computer programming. *Communications of the Association for Computing Machinery*, 12(10):576–580, 1969.
Z. Manna and A. Pnueli. Axiomatic approach to total correctness of programs. *Acta Inf.*, 3:243–263, 1974.

Burstall's proof method by hand-simulation and a little induction

- Program **do** odd(x) **and** x ≥ 3 → x := x+1
  □ even (x) **and** x ≥ 2 → x := x/2
  od
- Proof chart

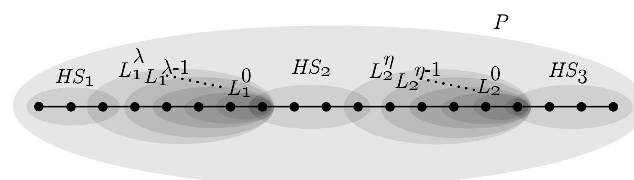


R. Burstall. Program proving as hand simulation with a little induction. *Information Processing*, 308–312. North-Holland, 1974.

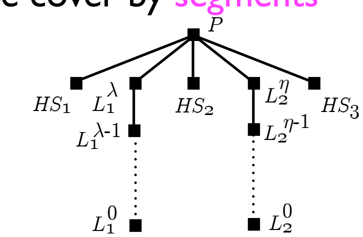
P. Cousot and R. Cousot. Sometime = always + recursion ≡ always, on the equivalence of the intermittent and invariant assertions methods for proving inevitability properties of programs. *Acta Informatica*, 24:1–31, 1987.

Burstall's proof method by hand-simulation and a little induction

- Iterative program but recursive proof structure

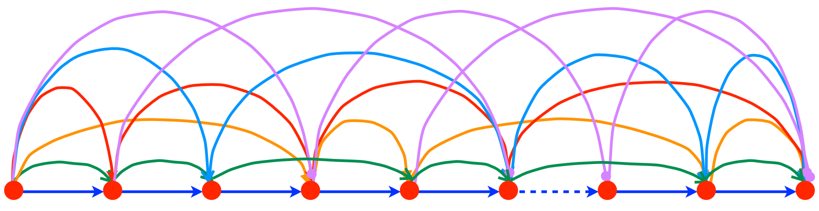


- Inductive trace cover by segments



Podelski-Rybalchenko

- Transition invariants are abstractions of trace segments covering the trace semantics by their extremities

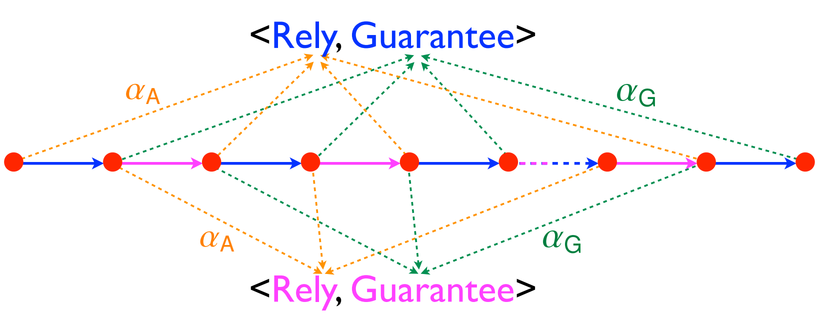


- Termination based on Ramsey theorem on colored edges of a complete graph, no recursive structure

A. Podelski and A. Rybalchenko. Transition invariants. *LICS*, 32–41, 2004.
F. P. Ramsey. On a problem of formal logic. In *Proc. London Math. Soc.*, volume 30, pages 264–285, 1930.

Rely-guarantee

- Example of abstraction of segments into rely-guarantee/contracts state properties:



Joey W. Coleman, Cliff B. Jones: A Structural Proof of the Soundness of Rely/guarantee Rules. J. Log. Comput. 17(4): 807-841 (2007)

Trace semantics segmentation

- Recursive trace segmentation

Definition 2. An *inductive trace segment cover* of a non-empty set $\chi \in \wp(\Sigma^{+\infty})$ of traces is a set $C \in \mathfrak{C}(\chi)$ of sequences S of members B of $\wp(\alpha^{+}(\chi))$ such that

1. if $S\,S' \in C$ then $S \in C$ (prefix-closure)
2. if $S \in C$ then $\exists S' : S =_{\chi} S'$ (root)
3. if $S\,BB' \in C$ then $B \ni B'$ (well-foundedness)
4. if $S\,BB' \in C$ then $B \subseteq \biguplus_{S\,BB' \in C} B'$ (cover). □

- Proof by induction on the possibly infinite but well-founded trace segmentation tree
- *Orthogonal* to proofs on segment sets (using variant functions, Ramsey theorem, etc.)

Conclusion

More in the paper

- The presentation was deliberately intended to be simple and intuitive
- The paper provides
  - More topics (e.g. abstract trace covers/proofs)
  - More technical details (e.g. fixpoint definitions of the various abstract termination semantics)
  - More examples (e.g. a more detailed piecewise linear termination abstraction)

Contributions

- Formalization of existing termination proof methods as abstract interpretations
- Pave the way for new *backward* termination static analysis methods (going beyond reduction of termination to safety analyzes)
- The new concept of trace semantics segmentation is not specific to termination and applies to all specification/verification/analysis methods

Future work

- Abstract domains for termination
- Semantic techniques for segmentation inference
- Eventuality verification/static analysis
- (General) liveness (*) verification/static analysis

(*) Beyond LTL, as defined in

Bowen Alpern, Fred B. Schneider: Defining Liveness. Inf. Process. Lett. (IPL) 21(4):181-185 (1985)2EEBowen Alpern, Fred B. Schneider: Defining Liveness. Inf. Process. Lett. (IPL) 21(4):181-185 (1985)

The end, thank you